E. J. Routh

E. J. Routh.
St. Peter's Coll.
1855
E. J. Routh.

PGT

[illegible]

[illegible]

[illegible]

Alexander Linwood

A TREATISE

ON

THE MOTION

OF

A SINGLE PARTICLE,

AND OF

TWO PARTICLES ACTING ON ONE ANOTHER.

By ARCHIBALD SANDEMAN, M.A.,
FELLOW AND TUTOR OF QUEENS' COLLEGE, CAMBRIDGE.

"All things move;
"The Sun flies forward to his brother Sun;
"The dark Earth follows wheel'd in her ellipse."

CAMBRIDGE: JOHN DEIGHTON.
LONDON: JOHN W. PARKER.

M.DCCC.L.

Prof. Alex. Ziwet
gt.
1-31-1923

Cambridge:
Printed at the University Press.

PREFACE.

THE following essay differs in some important respects from the treatises on the same subject which are in ordinary use.

I have throughout endeavoured to distinguish between the geometrical and mechanical, or, as they are sometimes called, the *kinematical* and *dynamical*, properties of motion. Properties of the former kind do not depend on physical causes, but arise solely from the manner in which the human mind necessarily conceives the relations of space and time, and, like all geometrical truths, require for their investigation no knowledge of the external world beyond what is necessary to develope the ideas of extension and succession: properties of the latter kind, on the contrary, depend entirely on the constitution of the physical universe, and can be discovered in no way other than by making experiments on material bodies. The course usually followed of presenting propositions relating to velocities and accelerations of motion in close connexion with (and even as forming parts of), propositions relating to forces, and the all-but-universal practice of calling accelerations of motion, accelerating forces, certainly do not tend to keep clear the distinction between the two kinds

of properties; yet, any confusion here will inevitably lead to incorrect conceptions of the relationships existing in the subject of motion. For the sake of clearness, I have kept the geometry separate from the mechanics of particle-motion; the term *force* has been applied exclusively to the *cause* of motional change, and the several effects of force have been spoken of as the statical, accelerating, and moving effects.

If the laws of motion be regarded as the simplest possible principles to which the relations existing between the motions of, and the forces acting on material, bodies can be reduced, particles of matter must be considered the only immediate subjects of the laws; for it is only in the case of an unextended body that complicated geometrical relations are avoided in representing the motion. The universal practice of deriving a knowledge of the motions of extended bodies by considering such bodies composed of unextended particles, confesses the necessity of regarding the fundamental laws as applicable to particles exclusively. In illustrating the laws of motion by appeals to direct experiment, I have therefore taken care to refer only to cases of motion in which the results would, apparently, be the same for particles as for the bodies under experiment. It is manifest that to propose, for instance, the motion of a wheel turning about a smooth axis, as direct illustration of the first law, or the motion of weights hung over a pully as direct example of the second law, cannot but tend to darken the views of a beginner. Every

case of motion, when thoroughly investigated by the methods which flow out of the fundamental laws, undoubtedly becomes a fresh proof of the truth of the laws; but no case of motion can be cited as direct proof in which the peculiarity of the motion depends on the fact of the extended dimensions of the moving body.

At the period of entering on the consideration of the laws of motion, we have the means of measuring motions and changes of motion, and we have from Statics the means of measuring forces. The object of the laws then becomes clear and definite; it is to connect the kinematical measures of motions and motional changes of particles, with the statical measures of the forces acting on the particles. Everything therefore with respect to measures is now fixed, except of course the mere optional units of space, time, and statical force. Such being the case, it is evidently incorrect to represent (as is frequently done), that the purpose of the laws is to seek for measures of force derived from the motions of bodies. In point of fact however the laws of motion do furnish such measures; but this is, as it were, accidental, and arises from the circumstance that the accelerations of motion produced by forces acting on a particle happen to be proportional to the measures of the forces which have been already adopted.

The form under which I have stated the three laws of motion is virtually the same as that under which they appear in Newton's *Principia*. The first Newtonian law

is universally adopted; but in a few lately published books the second is split up into two laws which respectively assert the proportionality of the statical and accelerating effects of forces with regard to the same particle, and the proportionality of the statical and moving effects of forces with regard to any particles; in these books the third Newtonian law, viz. the equality of action and reaction, is virtually ignored. Now, with reference to the breaking up of Newton's second law into two, it may be observed that if the law of the proportionality of the accelerating and statical effects of forces acting on a given particle be established, and if the constant of this proportionality be called the mass of the particle, the law of the proportionality of the moving and statical effects of forces acting on any particles, follows as an immediate consequence. These laws therefore are not essentially different, but each of them involves the other; the definition of mass is the only thing required to render them identical. If indeed a mechanical law be necessary in order to pass from the one to the other, it is that the constant of the proportionality of the statical to the accelerating effects of forces is not necessarily the same for different particles; but this is really only the most general supposition that can be made. The great objection to thus considering the second Newtonian law as involving two essentially different laws, is that so doing gives an appearance of appealing to some idea of mass derived independently of the law; whereas it is precisely to this very law that every department of

physics looks as the only source whence a definite idea of mass or quantity of matter can be derived.

In the books referred to, the law of the proportionality of the statical and moving effects of forces is represented to be the same as the law of the equality of action and reaction, but under a different form; in other words, Newton's second and third laws are represented to be identical. It is difficult to see how such a confusion of ideas could have arisen; the latter law compares the forces which two particles exert on one another (that is, the statical effect of the one with the statical effect of the other, or the moving effect of the one with the moving effect of the other), while the former law compares the statical with the moving effect of a force acting on a single particle. An indistinct perception of the principle —that all forces arise from the actions on one another of material bodies—may have had something to do in producing this strange jumbling together of essentially different laws.

It may be remarked that there is, strictly speaking, only one law of motion, viz., that which asserts the proportionality of the statical and accelerating effects of forces acting on a given particle. This law manifestly includes in it the law of uniformity of motion when no external forces act; the law that the constant of the proportionality of statical to accelerating effect of force is not necessarily the same for different particles—is a law of matter rather than of motion, since it connects force with matter rather

than with motion; and the law of the equality of action and reaction is a law of force rather than of motion, since it connects force with force and not force with motion. Even the principle called D'Alembert's is only a generalization of the second Newtonian law, since it hangs immediately on the principle that two systems of forces, which produce the same moving effects in a system of particles, are statically equivalent.

My chief aim in the following treatise has been to exhibit plainly the true fundamental relationships in the science of motion, by avoiding defects and errors such as have just been pointed out. It is scarcely possible to over-estimate the importance of a sound philosophy relative to the principles of motion, whether they be regarded as leading to valuable results in their own especial province, or as forming the groundwork of all physical science, or as furnishing rich materials for philosophizing on the nature of human knowledge.

TABLE OF CONTENTS.

CHAP. II. THE LAWS OF PARTICLE MOTION.

CHAP. III. THE FREE MOTION OF A PARTICLE.

CHAP. V. CONSTRAINED MOTION OF PARTICLES.

CHAP. VI. Resisted Motion of a Particle.

CHAP. VII. Disturbed Motion of a Particle.

THE

MOTION OF A SINGLE PARTICLE,

AND OF

TWO PARTICLES ACTING ON ONE ANOTHER.

ART. 1. EVERY change with regard to the states of motion and rest in material bodies is considered the effect of a cause which is called *force*. The science of motion has for its object to investigate the relations which exist between the motions of bodies and the forces acting on the bodies; so that when the motion of a body is known, the forces acting on it may be inferred, or when the forces are known the motion may be determined. Everywhere throughout the material universe phenomena of motion are presented; consequently the science of motion is of universal application in the physical sciences.

All bodies may be supposed to be made up of an infinite number of infinitely small parts, or particles; and the motion of a body may be considered completely determined only when the motion of each particle in the body is determined. Hence the natural order of proceeding towards a science of motion appears to be to begin with the motion of a single particle, to go on next to the motions of systems of particles, and so at last to arrive at the motions of bodies in general. In the following pages it is purposed to treat of the motion of a single particle, and of the motion of two particles acting on one another. For the sake of clearness there will be considered in order, (1) motion in itself separate from its cause, and the manner of measuring motions and changes of motion, (2) the mechanical laws which govern the motions of material particles, and (3) the application of the laws to various cases of motion.

CHAPTER I.

THE GEOMETRY OF A MOVING POINT'S MOTION.

---

2. A POINT is said to move when its position in space is continually changing; the line which passes through all its successive positions is its path; and its direction of motion at any point is indicated by the tangent drawn to the path at that point.

If, during its motion, a moving point pass over equal lengths of its path in all equal intervals of time, its motion is said to be uniform; and its speed, or *velocity*, is measured by the length of path passed over in an interval of time assumed as the unit. The unit of time generally adopted is a second, and velocity is generally expressed in feet: thus it is usual to say that a body moves at the rate of so many feet per second.

Let v denote the velocity of a point moving uniformly, and s the space described, or length of path passed over, by it in an interval of time denoted by t: since a space v is passed over by the point in each unit of time, a space vt is passed over by it in t units;

$$\therefore \; s = vt.$$

If the motion of a point be not uniform, the velocity at any point of its path is measured by the space which it would have described in an unit of time if, after passing the point, its motion had remained uniform.

Let s be the space described by a moving point in an interval of time t, beginning at a fixed epoch, and let v be the velocity of the point at the end of that interval. At the end of a greater interval of time $t + \delta t$, measured from the same epoch, let $s + \delta s$ be the space described. During the time δt the velocity of the point continually varies; but if v' be the

greatest velocity during this interval and $v_{\prime}$ the least, it is plain that

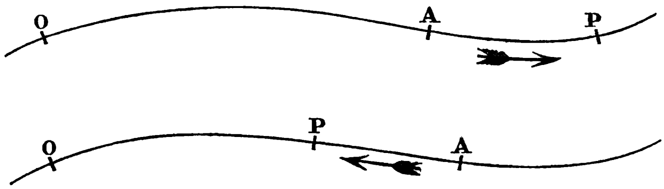
$$\delta s \text{ is not greater than } v'\delta t, \text{ and not less than } v_{\prime}\delta t;$$

$$\therefore \frac{\delta s}{\delta t} \text{ is not greater than } v', \text{ and not less than } v_{\prime}.$$

And this is true however small δt may be. But as δt continually diminishes and ultimately vanishes, each of the quantities v' and $v_{\prime}$ continually approaches and ultimately equals v. Hence, adopting the language of differentials,

$$\frac{ds}{dt} = v.$$

Suppose A to be the place of the moving point at the beginning of the time t, and P its place at the end, the point having described the length of path AP ($=s$) in the interval;



let O be a point assumed as an origin from which to measure distances along the path, and put $OA = a$, $OP = S$.

If the motion be in direction OA,

$$S = a + s;$$

$$\therefore \frac{dS}{dt} = \frac{ds}{dt} = v.$$

But if the motion be in direction AO,

$$S = a - s;$$

$$\therefore \frac{dS}{dt} = -\frac{ds}{dt} = -v.$$

This shews that if + prefixed to a velocity denote a motion in one direction, − prefixed will denote a motion in the opposite direction.

COR. If any quantity vary continuously in magnitude, and if q represent its magnitude at the end of the time t

measured from a fixed epoch, $\frac{dq}{dt}$ will represent its velocity of increase at the instant. If $\frac{dq}{dt}$ be negative, the velocity of increase is negative, that is, there is a velocity of decrease.

3. *Having given the path of a moving point, the law of its velocity, and its position at a given instant; to find its position at any other instant.*

Let a be the distance, from a fixed point in the path, of the position of the moving point at the given instant; and after a time t, measured from the given instant, let v be the velocity of the moving point, and s its distance from the fixed point—all distances being measured along the path.

If the velocity be given at every point of the path, that is, if v be given in terms of s; we have, by integrating the equation $\frac{ds}{dt} = v$,

$$\int \frac{ds}{v} = t + C,$$

C being a quantity independent of s and t. Now when $t = 0$, $s = a$, and $\therefore \int_a \frac{ds}{v} = C$. And when $t = T$, any time after the given instant, let $s = S$, $\therefore \int^S \frac{ds}{v} = T + C$.

$$\text{Hence } \int_a^S \frac{ds}{v} = T,$$

an equation which gives S in terms of T.

If however the velocity be given at every instant of the motion, that is, if v be given in terms of t; we have, by integrating the same equation,

$$s + C = \int v dt,$$

in which C is independent of s and t. When $t = 0$, $s = a$, $\therefore$ $a + C = \int_0 v dt$. And when $t = T$, $s = S$, $\therefore$ $S + C = \int^T v dt$.

$$\text{Hence } S - a = \int_0^T v dt,$$

which gives S in terms of T.

From one or other of these equations the position of the moving point may be found at any instant.

COR. *A point moves from rest in a given path, and its velocity at any instant is proportional to the time elapsed since its motion commenced; to find the space described by it in any given time.*

Let s denote the space described in the time t, measured from the beginning of the motion. The velocity at the end of this time may be represented by $v_{,}.t$, $v_{,}$ being the velocity at the end of a unit of time elapsed from the beginning of the motion; therefore

$$\frac{ds}{dt} = v_{,}t,$$

and integrating we get

$$s = \frac{1}{2} v_{,} t^2,$$

no constant being added because $s = 0$ when $t = 0$. This result shews that the space described in a given time, measured from the commencement of motion, is proportional to the square of the time. It also shews that the space is equal to half the product of the time into the final velocity: for $s = \frac{1}{2} . t . v_{,}t$, and $v_{,}t$ is the final velocity.

4. *The motion of a moving point at any point of its path may be supposed to arise from three co-existent velocities in different directions.*

Let three straight lines drawn through a fixed point in any different directions be taken for co-ordinate axes. At the end of a time t elapsed from a fixed epoch, let x, y, z be the co-ordinates of a moving point, and s the length of path intercepted between it and a fixed point in its path. In an indefinitely small time dt, the length of path s is increased by $\frac{ds}{dt}dt$, and the co-ordinates x, y, z are respectively increased by $\frac{dx}{dt}dt$, $\frac{dy}{dt}dt$, $\frac{dz}{dt}dt$. Hence the velocity $\frac{ds}{dt}$ of the moving point along its path may be supposed to arise

from simultaneous velocities of increase $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ of its co-ordinates, or in other words, from three co-existent velocities in different directions.

The velocities of increase of the co-ordinates of a moving point are usually called the component velocities of the point, and the velocity along the path is called the resultant velocity.

COR. 1. *If through a point three straight lines be drawn representing in magnitude and direction the component velocities of a moving point, then the diagonal of the parallelopiped formed on these lines as edges, passing through the same point, will represent the resultant velocity in magnitude and direction.*

For if δx, δy, δz be the increments of the co-ordinates of the moving point corresponding to an increment δs of the length of path s described by it, the chord of the arc δs is the diagonal of a parallelopiped whose edges are $\delta x, \delta y, \delta z$; and this is true however small δs may be. But if δs continually diminish and ultimately vanish, $\delta s, \delta x, \delta y, \delta z$ continually tend to be, and are ultimately, proportional to $\frac{ds}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$; and the arc δs, its chord, and the tangent to the path at the extremity of s continually approach and ultimately coincide with one another. Hence if through the moving point three straight lines be drawn parallel respectively to the axes of x, y, z, whose lengths are proportional to $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$; and if on these lines as edges a parallelopiped be formed, the direction of the diagonal, drawn through the moving point, will coincide with the tangent to the path, and its length will be proportional to $\frac{ds}{dt}$.

This proposition is sometimes cited as "the parallelopiped of velocities."

COR. 2. If a point move in a plane curve, its position may be referred to co-ordinate axes in the plane of motion;

and its velocity at any point of its path may be supposed to arise from two component velocities parallel to the co-ordinate axes; hence,

If through a point two straight lines be drawn representing in magnitude and direction the component velocities of a moving point, the diagonal, passing through the same point, of the parallelogram formed on the straight lines as edges, will represent the resultant velocity in magnitude and direction. This proposition is usually cited as "the parallelogram of velocities."

COR. 3. If u, v, w, denote the component velocities of a moving point parallel respectively to the co-ordinate axes, the velocity of the point is

$$= \sqrt{(u^2 + v^2 + w^2 + 2\alpha vw + 2\beta wu + 2\gamma uv)};$$

α, β, γ, being the cosines of the inclinations to one another of the co-ordinate axes. And the equations of a line, parallel to which the point is moving, are

$$\frac{x}{u} = \frac{y}{v} = \frac{z}{w}.$$

It is usual to take rectangular axes of reference; and then the resultant velocity is

$$= \sqrt{u^2 + v^2 + w^2};$$

and the direction of motion makes angles with the co-ordinate axes whose cosines are respectively

$$\frac{u}{\sqrt{u^2 + v^2 + w^2}}, \quad \frac{v}{\sqrt{u^2 + v^2 + w^2}}, \quad \frac{w}{\sqrt{u^2 + v^2 + w^2}}.$$

In order to simplify calculations the co-ordinate axes are always supposed to be at right angles to one another, unless the contrary is expressly stated; and the component velocities of a point parallel to rectangular axes, are usually called the velocities of the point in the directions of the respective axes. Hence the velocity of a point in any given direction is found by multiplying the velocity of the point into the cosine of the angle which the direction of motion makes with the given direction. Thus if u, v, w be the velocities of a moving point

in the directions of the co-ordinate axes, the velocity of the point in a direction making angles α, β, γ, with the axes is

$$= u \cos \alpha + v \cos \beta + w \cos \gamma.$$

COR. 4. Let the path of a moving point be a plane curve. Let r, θ be the polar co-ordinates of the point at the end of the time t measured from a fixed epoch; and let x, y be its co-ordinates referred to rectangular axes through the pole.

$$x = r \cos \theta, \qquad y = r \sin \theta;$$

$$\therefore \frac{dx}{dt} = \frac{dr}{dt} \cos \theta - r \frac{d\theta}{dt} \sin \theta, \quad \frac{dy}{dt} = \frac{dr}{dt} \sin \theta + r \frac{d\theta}{dt} \cos \theta.$$

Hence the velocity of the particle in direction of r

$$= \frac{dx}{dt} \cos \theta + \frac{dy}{dt} \sin \theta = \frac{dr}{dt}.$$

And the velocity in direction perpendicular to r

$$= \frac{dy}{dt} \cos \theta - \frac{dx}{dt} \sin \theta = r \frac{d\theta}{dt}.$$

Also the whole velocity

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2}.$$

These expressions for the radial and transversal velocities of a point may be found at once from the equation

$$\frac{dx}{dt} = \frac{dr}{dt} \cos \theta - r \frac{d\theta}{dt} \sin \theta.$$

For the position of the axis of x being perfectly arbitrary, provided only it be fixed, let it be so taken that at the time t, r is just coincident with it; then $\theta = 0$ at the instant, and

$$\therefore \text{ radial velocity} = \frac{dr}{dt}.$$

Again, let the position of the axis of x be so assumed that at the time t, r makes with it an angle $\frac{3\pi}{2}$; then putting

$\theta = \frac{3\pi}{2}$ in the above equation, we get

$$\text{transversal velocity} = r\frac{d\theta}{dt};$$

in the direction of increase of the angle vector. It is clear that $\frac{d\theta}{dt}$ is independent of the fixed line from which θ is measured.

5. *Having given the law of the component velocities of a moving point, and its position at a given instant; to find its position at any other instant, and the path which it describes.*

Let x, y, z be the co-ordinates of the moving point and u, v, w its component velocities at the end of the time t, measured from the given instant, and let a, b, c be its co-ordinates at the beginning of the time.

$$\text{We have } \frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w.$$

If u, v, w be given in terms of the time, from the first of the equations, we have

$$x + C = \int u dt,$$

C being independent of x and t. When $t = 0$, $x = a$; $\therefore$ $a + C = \int_0 u dt$; and when $t = t$, $x = x$; $\therefore x + C = \int^t u dt$. Hence

$$x - a = \int_0^t u dt.$$

In a similar way we get

$$y - b = \int_0^t v dt, \quad z - c = \int_0^t w dt.$$

These three equations give the position of the moving point at the end of the time t. And if t be eliminated between them, there will result two equations, which will be the equations of the path.

But if u, v, w be given in terms of x, y, z, from the same equations, by eliminating t, we have

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

From these two equations we must get by integration two other equations of the forms

$$f(x, y, z, C) = 0, \quad F(x, y, z, C') = 0;$$

in which C and C' are quantities independent of x, y, z. These are the equations of the path. Since the path passes through the point (a, b, c),

$$\therefore f(a, b, c, C) = 0, \quad F(a, b, c, C') = 0.$$

From which C and C' are known.

To find the position of the moving point at any instant we must find two of the quantities x, y, z in terms of the third from the equations of the path, and substitute the expressions found in one of the original equations. The result will then be an equation involving t and one only of the quantities x, y, z. By integrating and correcting we shall find one of the co-ordinates of the moving point in terms of the time; the other co-ordinates will then be known from the equations of the path.

Cor. If each of the component velocities be constant

$$x - a = ut, \quad y - b = vt, \quad z - c = wt;$$

$\therefore$ the equations of the path are

$$\frac{x-a}{u} = \frac{y-b}{v} = \frac{z-c}{w}.$$

Hence the point moves in a straight line.

If again the component velocities be to each other in constant ratios, the path is also a straight line. For let

$$u : v : w = \alpha : \beta : \gamma, \quad \alpha, \beta, \gamma, \text{ being invariable,}$$

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{\gamma}.$$

Integrating, and applying the proper constants,

$$\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma}.$$

6. When the velocity with which a moving point moves along its path varies continually from instant to instant, the motion is said to be *accelerated* or *retarded* according as the velocity is increasing or decreasing.

If, during the motion, the velocity receive equal increments in all equal intervals of time, the motion is said to be uniformly accelerated; and the *acceleration along the path* is measured by the increment of velocity received in a unit of time.

Let a denote the acceleration of motion along its path of a moving point, whose motion is uniformly accelerated. Since in each unit of time during the motion, the velocity receives an increase equal to a, therefore in any interval of time t the velocity receives an increase equal to at. And hence if v denote the velocity of the moving point at the beginning of the time t, and v' the velocity at the end of the same time,

$$v' - v = at.$$

Similarly, if, during the motion, the velocity suffer equal decrements in all equal intervals of time, the motion is said to be uniformly retarded; and the retardation along the path is measured by the decrement of velocity in a unit of time. If ρ denote the retardation along the path, and v, v' the velocities of the moving point at the beginning and end of an interval of time t,

$$v - v' = \rho t, \quad \text{or} \quad v' - v = (-\rho)t,$$

which indicates the same change of velocity as if the motion had been uniformly accelerated, and the acceleration along the path equal to $(-\rho)$. Consequently retarded motions will be included in accelerated motions, if negative accelerations be admitted as well as positive accelerations. It will save trouble to understand accelerated motion in this extended sense.

If the motion of a moving point be not uniformly accelerated, the acceleration along the path at any instant is measured by the increment of velocity which would have been received in a unit of time, if the motion after the instant had remained uniformly accelerated.

Let v be the velocity of a moving point along its path at the end of the time t, measured from a fixed epoch; and after a greater time $t + \delta t$, measured from the same epoch, let $v + \delta v$ be the velocity:—also let a be the acceleration along the path at the end of the time t. During the interval of time δt the acceleration along the path is continually varying; but if a' be the greatest acceleration along the path during this interval, and $a_{,}$ the least, it is clear that

δv is not greater than $a'\delta t$, and not less than $a_{,}\delta t$;

Therefore

$\frac{\delta v}{\delta t}$ is not greater than a', and not less than $a_{,}$.

And this is true however small δt may be. But when δt continually diminishes and ultimately vanishes, each of the quantities a', $a_{,}$, continually approaches to, and ultimately equals a,

$$\therefore \frac{dv}{dt} = a.$$

Let s be the length of path intercepted between the position of the moving point at the end of the time t and a fixed point in the path. Then $v = \frac{ds}{dt}$, and therefore

$$\frac{d\left(\frac{ds}{dt}\right)}{dt} = a, \text{ or } \frac{dtd^2s - dsd^2t}{dt^3} = a,$$

or, if t be considered the independent variable,

$$\frac{d^2s}{dt^2} = a.$$

Again, if v be a function of s, then $\frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\frac{dv}{ds}$;

$$\therefore v\frac{dv}{ds} = a.$$

7. *Having given the path of a moving point and the law of acceleration along the path, also the position and velocity of the*

point at a given instant; to find its position and velocity at any other instant.

Let s be the length of path intercepted between the given position and the position of the moving point after a time t measured from the given instant; a the acceleration along the path after the same time; and V the velocity of the point at the given instant. We have

$$\frac{d^2s}{dt^2} = a.$$

If a be given in terms of t; then by integrating,

$$\frac{ds}{dt} + C = \int a\,dt.$$

Now when $t = 0$, $\frac{ds}{dt} = V$, therefore,

$$\frac{ds}{dt} = V + \int_0^t a\,dt.$$

And, again integrating,

$$s + C' = \int\left(V + \int_0^t a\,dt\right)dt.$$

When $t = 0$, $s = 0$. Hence

$$s = \int_0^t\left(V + \int_0^t a\,dt\right)dt$$

$$= Vt + \int_0^t\left(\int_0^t a\,dt\right)dt.$$

But, if a be given in terms of s, multiplying each side of the first equation by $2\frac{ds}{dt}$ and integrating, we get

$$\left(\frac{ds}{dt}\right)^2 + C = 2\int a\,ds.$$

When $s = 0$, $\frac{ds}{dt} = V$; therefore

$$\left(\frac{ds}{dt}\right)^2 = V^2 + 2\int_0^s a\,ds,$$

$$\text{and } \frac{ds}{\sqrt{V^2 + 2\int_0^s a\,ds}} = dt.$$

In order to have the complete solution of the question, both signs of the radical must be considered. It will generally, however, be most convenient to begin by taking the sign which is the same as the sign of V; and then in following the motion from $t = 0$ onwards, the sign can only change when the quantity under the radical becomes zero, that is, when the velocity becomes zero. Immediately after this takes place, the sign will be the same as that of a.

Integrating again, and correcting by making s and t vanish together,

$$\int_0^s \frac{ds}{\sqrt{V^2 + 2\int_0^s a\,ds}} = t.$$

From one or other of these equations the position and velocity of the moving point are known at any instant.

COR. *If the motion of a point begin from rest and be uniformly accelerated, the velocity after any time from the beginning of the motion will vary as the time, and the space described as the square of the time.*

$$\text{For in this case, } \frac{d^2s}{dt^2} = a, \text{ a constant,}$$

$$\therefore \frac{ds}{dt} = at,$$

no constant being needed since $\frac{ds}{dt}$ and t begin together,

$$\therefore s = \tfrac{1}{2}at^2$$

since $s = 0$, when $t = 0$.

Hence also the spaces described by the point in any equal successive intervals of time, beginning at the commencement of the motion, are proportional to

$$1^2,\ 2^2 - 1^2,\ 3^2 - 2^2,\ 4^2 - 3^2,\ \&c.,$$

that is, to the successive odd numbers, 1, 3, 5, 7, &c.

If at the beginning of the time t the point had a velocity V along its path, instead of being at rest, the first integral would be

$$\frac{ds}{dt} = V + at,$$

and the second integration then gives

$$s = Vt + \tfrac{1}{2}at^2.$$

If the time be divided into equal intervals each equal to τ, and if s_1, s_2, s_3, &c. denote the spaces described in successive intervals beginning at the beginning of the line t,

$$s = V\tau + \tfrac{1}{2}a\tau^2, s_1 + s_2 = V.2\tau + \tfrac{1}{2}a(2\tau)^2, s_1 + s_2 + s_3 = V.3\tau + \tfrac{1}{2}a(3\tau)^2, \&c.$$

$$\therefore s_1 = \tfrac{1}{2}a\tau^2\left(\frac{2V}{a\tau} + 1\right), s_2 = \tfrac{1}{2}a\tau^2\left(\frac{2V}{a\tau} + 3\right), s_3 = \tfrac{1}{2}a\tau^2\left(\frac{2V}{a\tau} + 5\right), \&c.$$

$$\text{And generally } s_n = \tfrac{1}{2}a\tau^2\left\{\frac{2V}{a\tau} + (2n-1)\right\}$$

or the spaces described in any successive intervals of time during the motion are proportional to successive terms of an arithmetic series whose common difference is two.

8. If the position of a moving point at any instant of its motion be referred to three fixed straight lines drawn through a point in different directions, the accelerations of the motions of increase of the co-ordinates are called the component accelerations of the motion of the moving point parallel respectively to the co-ordinate axes. Let x, y, z be the co-ordinates of the moving point at the end of the time t measured from a fixed epoch, and let α, β, γ be the component accelerations of the point parallel respectively to the axes of x, y, z at the end of the same time. We have

$$\frac{d^2x}{dt^2} = \alpha, \quad \frac{d^2y}{dt^2} = \beta, \quad \frac{d^2z}{dt^2} = \gamma.$$

The resultant acceleration at the instant is the acceleration of the point's motion along the path which it would begin to describe if it had no velocity but began at the instant to move from rest, and the direction of the resultant is the direction in which the point would begin to move.

If the moving point began to move from rest at the point (x, y, z), the increments of x, y, z in the infinitely small time dt would be respectively (cor. of last article)

$$\frac{\alpha dt^2}{2}, \frac{\beta dt^2}{2}, \frac{\gamma dt^2}{2}.$$

And the space described would therefore be

$$= \tfrac{1}{2} dt^2 \sqrt{\alpha^2 + \beta^2 + \gamma^2 + 2a\beta\gamma + 2b\gamma\alpha + 2c\alpha\beta}$$

(a, b, c being the cosines of the angles which the co-ordinate axes make with one another). Hence the resultant acceleration is

$$= \sqrt{\alpha^2 + \beta^2 + \gamma^2 + 2a\beta\gamma + 2b\gamma\alpha + 2c\alpha\beta},$$

and its direction is parallel to the line whose equations are

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}.$$

It follows from this that if the sides of a parallelopiped represent the component accelerations of a motion in magnitude and direction, the diagonal will represent the resultant acceleration in magnitude and direction.

For the sake of simplicity the component accelerations of the motion of a moving point are always supposed (unless the contrary be expressly stated) to be parallel to three straight lines at right angles to one another; and the component acceleration parallel to any one of the straight lines is called the acceleration in the direction of that line. Hence if α, β, γ be the accelerations in directions of three rectangular axes, the resultant acceleration is

$$= \sqrt{\alpha^2 + \beta^2 + \gamma^2},$$

and the acceleration in any other direction which makes angles A, B, C with the co-ordinate axes is

$$= \alpha \cos A + \beta \cos B + \gamma \cos C,$$

or, the acceleration in any given direction is equal to the product of the resultant acceleration into the cosine of the angle which the direction of the resultant makes with the given direction.

p— COR. 1. *Any motion whatever of a moving point may, at any point of the path, be supposed to arise from a velocity along the tangent and two component accelerations, one along the tangent and the other perpendicular to the tangent in the osculating plane.*

Let x, y, z be the co-ordinates of a moving point at the end of the time t measured from a fixed epoch; and let s be the length of path intercepted between $(x\ y\ z)$ and a fixed point in the path.

$$\frac{dx}{dt}=\frac{ds}{dt}\cdot\frac{dx}{ds}.$$

Therefore $$\frac{d^2x}{dt^2}=\frac{d^2s}{dt^2}\cdot\frac{dx}{ds}+\left(\frac{ds}{dt}\right)^2\frac{d^2x}{ds^2}.$$

Similarly, $$\frac{d^2y}{dt^2}=\frac{d^2s}{dt^2}\cdot\frac{dy}{ds}+\left(\frac{ds}{dt}\right)^2\frac{d^2y}{ds^2},$$

and $$\frac{d^2z}{dt^2}=\frac{d^2s}{dt^2}\cdot\frac{dz}{ds}+\left(\frac{ds}{dt}\right)^2\frac{d^2z}{ds^2}.$$

Multiplying the first of these equations by $\frac{dx}{ds}$, the second by $\frac{dy}{ds}$, the third by $\frac{dz}{ds}$, and adding, we get

$$\left\{\text{since } \left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1,\right.$$

$$\left.\text{and } \therefore\ \frac{dx}{ds}\frac{d^2x}{ds^2}+\frac{dy}{ds}\cdot\frac{d^2y}{ds^2}+\frac{dz}{ds}\cdot\frac{d^2z}{ds^2}=0\right\},$$

$$\frac{d^2x}{dt^2}\cdot\frac{dx}{ds}+\frac{d^2y}{dt^2}\cdot\frac{dy}{ds}+\frac{d^2z}{dt^2}\cdot\frac{dz}{ds}=\frac{d^2s}{dt^2},$$

which is the acceleration along the tangent.

Again, multiplying the same equations in order by $\frac{d^2x}{ds^2}$, $\frac{d^2y}{ds^2}$, $\frac{d^2z}{ds^2}$, and adding, we have

$$\frac{d^2x}{dt^2}\frac{d^2x}{ds^2}+\frac{d^2y}{dt^2}\frac{d^2y}{ds^2}+\frac{d^2z}{dt^2}\frac{d^2z}{ds^2}$$

$$=\left\{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2\right\}\left(\frac{ds}{dt}\right)^2.$$

But if ρ be the radius of absolute curvature of the path at the point (x, y, z),

$$\frac{1}{\rho^2}=\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2,$$

and the cosines of the angles which ρ makes with the co-ordinate axes are $\rho\dfrac{d^2x}{ds^2}$, $\rho\dfrac{d^2y}{ds^2}$, $\rho\dfrac{d^2z}{ds^2}$ respectively. Therefore the acceleration in direction of ρ is

$$=\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2.$$

Lastly, from the same three equations we have

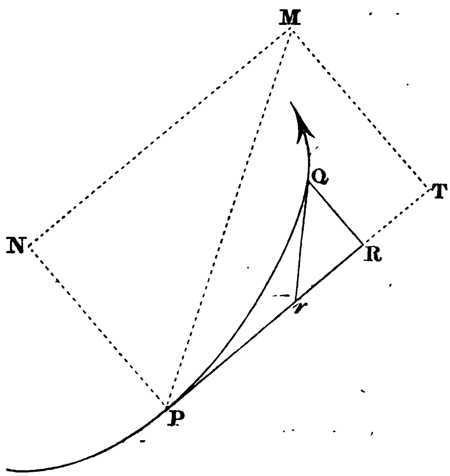
$$\frac{d^2x}{dt^2}\left(\frac{dy}{ds}\frac{d^2z}{ds^2}-\frac{dz}{ds}\frac{d^2y}{ds^2}\right)+\frac{d^2y}{dt^2}\left(\frac{dz}{ds}\frac{d^2x}{ds^2}-\frac{dx}{ds}\frac{d^2z}{ds^2}\right)$$

$$+\frac{d^2z}{dt^2}\left(\frac{dx}{ds}\frac{d^2y}{ds^2}-\frac{dy}{ds}\frac{d^2x}{ds^2}\right)=0,$$

therefore the acceleration perpendicular to the osculating plane $=0$. Hence the motion is compounded of a velocity $\dfrac{ds}{dt}$ along the path, an acceleration $\dfrac{d^2s}{dt^2}$ along the tangent or path, and an acceleration $\dfrac{1}{\rho}\left(\dfrac{ds}{dt}\right)^2$ along a normal to the path drawn in the osculating plane on the concave side.

The result arrived at may be illustrated thus:—Let PQ be the length of path described by the moving point in the small time δt. Draw the tangent PT, and the normal PN in the osculating plane at P, proportional to the accelerations in their respective directions (T and N suppose). Complete the rectangle NT, and draw the diagonal PM, which

is therefore in the direction of, and proportional to, the resultant acceleration. Draw QR perpendicular to PT. And let Pr be the space which the moving point would describe in the time δt if its motion were not accelerated (viz. $V\delta t$ if V be the velocity at P). Join Qr.



Now RQ is the space due to the normal acceleration, and rR is the space due to the tangential acceleration. And ultimately, when δt continually diminishes and ultimately vanishes,

$$RQ = \tfrac{1}{2} N . \delta t^2, \qquad rR = \tfrac{1}{2} T . \delta t^2, \qquad \text{(art. 7, cor.)}$$

$$\therefore RQ : rR = N : T = MT : TP.$$

Hence the right-angled triangles QRr, MTP are ultimately similar.

The lines QR, RP are ultimately in the osculating plane at P; consequently rQ is ultimately parallel to PM, that is, is ultimately in the direction of the resultant acceleration, and its length is the space due to the resultant acceleration.

From this we have

$$\text{tangential acceleration} = 2 \text{ limit of } \left(\frac{rR}{\delta t^2}\right)$$

$$= 2 \text{ limit of } \left(\frac{PR}{\delta t^2} - \frac{V}{\delta t}\right);$$

$$\text{normal acceleration} = 2 \text{ limit of } \left(\frac{RQ}{\delta t^2}\right);$$

$$\text{resultant acceleration} = 2 \text{ limit of } \frac{\sqrt{RQ^2 + (PR - V . \delta t)^2}}{\delta t^2};$$

and, tangent of the angle which the resultant acceleration makes with the tangent is

$$= \text{limit of } \left(\frac{RQ}{PR - V.\delta t}\right).$$

It is easily seen that the expressions for the tangential and normal accelerations are equivalent to those found above. For

$$PR = \frac{ds}{dt}.\delta t + \frac{d^2 s}{dt^2}.\frac{\delta t^2}{1.2} + \frac{d^3 s}{dt^3}.\frac{\delta t^3}{1.2.3} + \ldots$$

$$= V\delta t + \frac{d^2 s}{dt^2}.\frac{\delta t^2}{2} + \ldots$$

$$\therefore 2 \text{ limit of } \left(\frac{PR}{\delta t^2} - \frac{V}{\delta t}\right) = 2 \text{ limit of } \left(\frac{d^2 s}{dt^2}.\frac{1}{2} + \frac{d^3 s}{dt^3}\frac{\delta t}{1.2.3} + \ldots\right)$$

$$= \frac{d^2 s}{dt^2}.$$

$$\text{And } 2 \text{ limit of } \left(\frac{RQ}{\delta t^2}\right) = 2 \text{ limit of } \left\{\frac{RQ}{PR^2}.\left(\frac{PR}{\delta t}\right)^2\right\}.$$

$$\text{But } 2 \text{ limit of } \left(\frac{RQ}{PR^2}\right) = \frac{1}{\rho};$$

$$\text{and limit of } \left(\frac{PR}{\delta t}\right) = V;$$

$$\therefore 2 \text{ limit of } \left(\frac{RQ}{\delta t^2}\right) = \frac{V^2}{\rho}.$$

COR. 2. *If the path of a moving point be a plane curve and the position of the point at any instant be determined by polar co-ordinates; to find expressions for the accelerations along and perpendicular to the radius vector.*

Let r, θ be the polar co-ordinates of the moving point at the end of the time t, measured from a fixed epoch, and let x be the orthogonal projection of r on the prime radius.

$$x = r \cos\theta;$$

$$\therefore \frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt};$$

$$\text{and } \frac{d^2 x}{dt^2} = \left\{\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\cos\theta - \left(2\frac{dr}{dt}.\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\sin\theta.$$

This is an expression for the acceleration in direction of the

prime radius. Now the position of the prime radius may be assumed any whatever, provided only it be fixed. Let then its position be such that at the end of the time t, r is coincident with it;—and in the above expression put $\theta = 0$,

$$\therefore \text{ acceleration in direction of } r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2.$$

It is clear that $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ are the same whatever fixed line be taken for the prime radius.

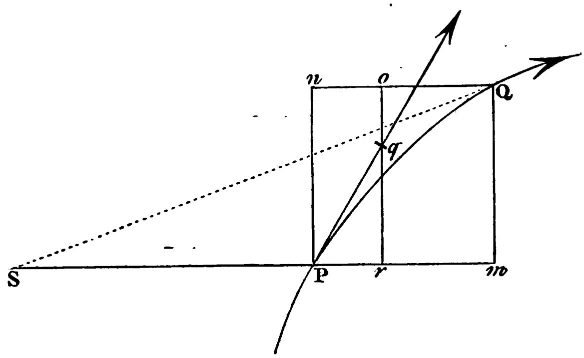
i.e.

Again, let the position of the prime radius be such that at the end of the time t, r is inclined to it at an angle $\frac{3\pi}{2}$, and we have

Acceleration perpendicular to r, in direction of increase of the angle vector, $= 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).$$

These expressions may also be found as follows:—Let PQ be the path described in the small time δt. Join P



and Q with the pole S; and on SP, produced if necessary, let fall the perpendicular Qm; also complete the rectangle mn. Draw Pq the tangent at P; and let Pq be the space which the moving point would describe in the time δt, if it moved uniformly with its velocity at P. Through q, draw oqr perpendicular to Pm or nQ.

It is manifest that rm is the space due to the acceleration in direction SP, and that qo is the space due to the acceleration in direction Pn. Hence when δt continually diminishes and ultimately vanishes, we have ultimately

$$rm = \tfrac{1}{2} R \,.\, \delta t^2, \; qo = \tfrac{1}{2} T \,.\, \delta t^2,$$

R and T denoting the radial and transversal accelerations.

$$\therefore\; R = 2 \text{ limit of } \left(\frac{Pm - Pr}{\delta t^2}\right),$$

$$T = 2 \text{ limit of } \left(\frac{Qm - qr}{\delta t^2}\right).$$

Now $Pr = \dfrac{dr}{dt} \,.\, \delta t$, and $qr = r \dfrac{d\theta}{dt} \,.\, \delta t$, (art. 4. cor. 4)

$$Pm = SQ \cos QSm - SP$$

$$= \left(r + \frac{dr}{dt} . \delta t + \frac{d^2 r}{dt^2} . \frac{\delta t^2}{2} + \dots\right) \cos \left(\frac{d\theta}{dt} \delta t + \frac{d^2\theta}{dt^2} . \frac{\delta t^2}{2} + \dots\right) - r$$

$$= \frac{dr}{dt} \delta t + \left\{\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt}\right)^2\right\} \frac{\delta t^2}{2} + \text{terms in } \delta t^3, \text{ \&c.}$$

And $Qm = SQ \sin QSm$

$$= \left(r + \frac{dr}{dt} \delta t + \frac{d^2 r}{dt^2} . \frac{\delta t^2}{2} \dots\right) \sin \left(\frac{d\theta}{dt} \delta t + \frac{d^2\theta}{dt^2} . \frac{\delta t^2}{2} \dots\right)$$

$$= r \frac{d\theta}{dt} . \delta t + \left(2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2}\right) \frac{\delta t^2}{2} + \text{terms in } \delta t^3, \text{ \&c.}$$

Substituting and proceeding to the limits, we get

$$R = \frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt}\right)^2, \qquad T = 2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2}.$$

If the point instead of moving in a plane move anyhow in space, and if r, θ be the polar co-ordinates of the orthogonal projection of the point on a fixed plane; the acceleration of motion may be resolved into a component acceleration perpendicular to this plane and two component accelerations parallel to the plane, respectively parallel and perpendicular to r. The components parallel to the plane will still be denoted by the above expressions.

COR. 3. If at the time t, r denote the distance of a moving point from a fixed point O, θ the angle which r makes with a fixed line Oz drawn through O, and ϕ the angle which rOz makes with a fixed plane passing through Oz, we can, by help of the last cor., find expressions for the component accelerations of the point's motion along r, perpendicular to r in the plane rOz, and in direction perpendicular to the plane rOz. Denoting these components by ρ, σ, τ respectively, we have (since $r \sin\theta$ and ϕ are the polar co-ordinates of the projection on a plane through O perpendicular to Oz of the moving point)

$$\rho \sin\theta + \sigma\cos\theta = \frac{d^2}{dt^2}(r\sin\theta) - r\sin\theta\left(\frac{d\phi}{dt}\right)^2,$$

$$\rho\cos\theta - \sigma\sin\theta = \frac{d^2}{dt^2}(r\cos\theta)$$

$$\tau = \frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\sin^2\theta\frac{d\phi}{dt}\right).$$

From which we get immediately,

$$\rho = \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2,$$

$$\sigma = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - r\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2,$$

$$\tau = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right)\sin\theta + 2r\frac{d\theta}{dt}\frac{d\phi}{dt}\cos\theta.$$

9. *Having given the position, velocity, and direction of motion of a moving point at a given instant, and also the law of the component accelerations of its motion; to find its path and its position at any instant.*

Let a, b, c be the co-ordinates of the given position; and let u, v, w be the component velocities of the moving point at the instant when it occupies that position, which are given since the velocity and direction of motion are given. Let x, y, z be the co-ordinates of the moving point at the end of the time t, measured from the given instant; and let α, β, γ

be the component accelerations of its motion at the end of the same time. We have

$$\frac{d^2x}{dt^2} = \alpha, \quad \frac{d^2y}{dt^2} = \beta, \quad \frac{d^2z}{dt^2} = \gamma.$$

If α, β, γ be given in terms of t, we get by integrating the first equation

$$\frac{dx}{dt} + C = \int \alpha dt,$$

$$\text{when } t = 0, \quad \frac{dx}{dt} = u,$$

$$\therefore \quad \frac{dx}{dt} = u + \int_0^t \alpha dt.$$

Integrating again, and making $x = a$ when $t = 0$,

$$x = a + \int_0^t \left(u + \int_0^t \alpha dt\right) dt$$

$$= a + ut + \int_0^t \left(\int_0^t \alpha dt\right) dt.$$

In a similar way we should get

$$y = b + vt + \int_0^t \left(\int_0^t \beta dt\right) dt, \quad z = c + wt + \int_0^t \left(\int_0^t \gamma dt\right) dt.$$

These equations give the co-ordinates of the moving point at any instant, and by eliminating t from them, there will result two equations, which will be the equations of the path. The component velocities at the end of any time t are respectively

$$u + \int_0^t \alpha dt, \quad v + \int_0^t \beta dt, \quad w + \int_0^t \gamma dt.$$

The motion of the point is thus completely determined.

If one or more of the quantities α, β, γ be given in terms of $x\ y\ z$, the general process cannot be shewn explicitly. In particular cases various artifices will suggest themselves, the object being either before or after integration to obtain two equations free from t which will be the equations of the path, and then the co-ordinates after any time may be found.

CHAPTER II.

THE FUNDAMENTAL LAWS OF MOTION OF MATERIAL PARTICLES.

---

10. The relations which exist between the motions of bodies and the forces that act on the bodies depend on principles which are capable of being expressed in the form of simple laws. But it is impossible immediately to establish these laws. For the motions of bodies which can be subjected to direct experiment take place under circumstances, so many, so various, and so inseparably connected with one another, that it is impossible to assign to each circumstance its due influence. In fact experiments in which the conditions of the laws are perfectly fulfilled cannot be made; and, in experiments which do not fulfil the conditions, it is impossible to take account of the disturbing influences without assuming the laws themselves. For instance, the motion of an extended body is perfectly known only when the motion of every point in it is known; and hence arises the necessity of considering first of all the motions of particles or bodies whose dimensions are infinitely small. But we have no immediate knowledge of such bodies, and therefore cannot immediately make experiments on them. Here then is a difficulty at the outset—that the bodies which must necessarily be the subjects of the laws of motion cannot from their nature be made the subjects of direct experiment.

Such being the case, the laws of motion do not admit of an immediate proof. The only way in which they can be arrived at seems to be this:—

(1) By making experiments in which the conditions specified by the several laws are approximately realized, it is found that the results specified by the laws are distinctly indicated as those to which the results of the experiments

tend more or less, according as the conditions are more or less nearly fulfilled.

(2) By investigating and calculating, according to the laws, all actual cases of motion, it is found that the results of the calculations uniformly agree with the results actually observed.

The laws of motion then are to be received because they are found adequate to explain fully and accurately all observed phenomena of motion.

The laws of motion are concerned with the motions of particles and the forces which act on the particles. A particle is a body whose dimensions in all directions are infinitely small; it is in short a material point. Motions of particles are known and measured by their velocities and accelerations. And forces (up to this point) are known only by their statical effects, and consequently are measured by them. Thus, a force, which, acting at a certain point of a body in a certain direction, balances a certain system of forces acting on the body, being adopted as a unit force; any force is measured by the number of unit forces which, acting together at a point of a body in a particular direction, balance the same system of forces as the force would, were it to act on the body at the same point in the same direction.

11. If a material particle be acted on by no external forces, or by external forces which balance one another, the laws relating to it are the following.

(1) *If the particle be at rest it will remain at rest.*

Constant experience shews that in order to move bodies from place to place, external force must be used;—and this, not only when known forces oppose their motion, but also when no such forces are sensible. Besides, there seems to be no reason why a particle should of itself begin to move in any one direction rather than in any other.

(2) *If the particle be in motion its path is a straight line.*

Experience shews that in all cases of curvilineal motion external forces are at work. If a stone be thrown in a direc-

tion inclined to the vertical, its path is concave towards the earth's surface; but this may be attributed to the force which draws the stone towards the earth in the vertical direction. Such a supposition is confirmed by observing that the stone's path is in a vertical plane; and that if the stone be thrown vertically upwards its path is a straight line:—thus giving ground for the inference that when no external force acts out of the direction of motion, the direction in which a body moves continues unchanged. If the stone be thrown along a horizontal plane, the force drawing it to the earth is counteracted, and its path is very nearly straight: and any deviations from perfect straightness may be attributed to the action of friction on the parts of the stone in contact with the plane; which supposition is rendered probable by the fact that the smoother the plane is, the more nearly straight is the path of the stone.

If a carriage in motion turn towards the right hand, a person in it feels a tendency towards the left hand side of the carriage; thus shewing that his body tends to continue its motion in the original direction.

There seems to be no reason why a particle moving in a direction should of itself tend to deviate from that direction on any one side rather than on any other.

(3) *If the particle be in motion its velocity remains constant.*

If a stone be thrown along a horizontal plane, it moves slower and slower until at length it comes to rest; but this may arise from the force of friction of the plane acting to destroy the motion. The supposition is rendered probable by observing that when the plane is made more smooth, and the friction thus diminished, the velocity is diminished by slower degrees, and the stone continues a longer time in motion.

If a carriage in motion be suddenly stopped, a person in it feels a tendency to be thrown towards the front; thus shewing that his body tends to preserve its original motion.

Experience testifies that whenever a motion is destroyed external force has acted.

The foregoing properties are summed up in the following law of motion, which expresses the perfect indifference of matter as to motion or rest, and implies that there are no internal forces in a material particle which, acting by themselves, tend to originate, modify, or destroy motion.

FIRST LAW OF MOTION. *If a particle be acted on by no external forces, or by external forces which balance one another, either it is at rest, or it moves in a straight line with a constant velocity.*

12. The laws which regulate the action of forces on a material particle may be stated as follows:

(1) *If a force act constantly, with the same intensity and in the same direction, on a particle which is initially either at rest or moving in the direction of the force, the resulting motion is wholly in direction of the force and is uniformly accelerated.*

At all small heights above a given place on the earth's surface, the weight of a heavy body is sensibly the same, and acts in the same direction: here then is the case of a force acting constantly with the same intensity in the same direction. Hence, if the law stated be true, the motion of a heavy body dropped from rest or thrown vertically upwards or downwards ought to be uniformly accelerated. This may be tested by observing whether the motion has the characteristics of a uniformly accelerated motion, such as the velocity after any time from the beginning of motion being proportional to the time, and the space described proportional to the square of the time, or the spaces described in equal successive intervals of time being proportional to successive odd numbers. (art. 7. cor.) It is found that the motions of heavy bodies let fall from rest, or thrown vertically upwards or downwards, are very nearly uniformly accelerated, and that their paths are vertical straight lines. Deviations from this result are accounted for by the resistance which the air offers to the motion of heavy bodies; and there seems to be no doubt that if experiments could be performed in a perfect vacuum, the motions would be found to be uniformly accelerated and the paths straight.

The motions of heavy bodies directly up and down in-

clined planes might also be experimented on in order to test the truth of the law; for the force which urges a body down a plane is the same at all points in the plane, being to the weight of the body in the same ratio as the sine of the plane's inclination to the horizon is to unity. But in an experiment of this kind the friction of the plane (for no plane can be found perfectly smooth) will interfere with the result as well as the resistance of the air.

From this law it follows that the acceleration of motion produced by a constant force acting in the direction of motion is the same at every instant of the motion; and therefore the accelerating effect of a constant force on a particle moving in its direction is wholly independent of the velocity of the particle in that direction.

(2) *The accelerations of motion produced by different forces of constant intensities, when they act singly on a given particle in the direction of its motion, are in the directions and proportional to the intensities of the forces.*

Let a heavy body of weight ω be placed on a smooth plane which is inclined to the horizon at an angle i; and when the body is let fall from a point in the plane or projected directly up or down it, let a be the acceleration of its motion down the plane: then the intensity of the force down the plane, which produces the acceleration of motion a, is $= \omega \sin i$. When the same body is dropped down another plane, whose inclination to the horizon is i', or projected directly up or down it, let a' be the acceleration of motion down it; then the force whose intensity is $\omega \sin i'$ produces the acceleration a'. Therefore if the law be true we must have

$$a : a' = \omega \sin i : \omega \sin i' = \sin i : \sin i'.$$

In making an experiment to test the correctness of this result, the friction of the planes and the resistance of the air will interfere. But these may be diminished very considerably by having the planes as smooth as possible, and by making them incline to the horizon at small angles.

From this law there follows as a consequence the more general law:

(3) *The accelerations of motion produced by any forces, acting singly on a given particle in the direction of its motion, are in the directions and proportional to the intensities of the forces.*

No force can produce all of a sudden a finite change of velocity in the motion of a particle. If then the intensity of a force vary from instant to instant, its accelerating effect at any instant can be measured only by the acceleration which it would produce if during some interval of time after the instant it had continued to act with the same intensity and in the same direction as at the instant. But in such case (by the foregoing laws) the motion during the interval would have been uniformly accelerated, and the acceleration proportional to the intensity of the force. Now the acceleration of the motion at the instant is measured by the acceleration which would have been, had the motion immediately after the instant continued uniformly accelerated (art. 6). Hence the accelerating effects of forces, acting on a given particle in the direction of its motion, are proportional to the statical effects of the forces.

It is to be remembered that in all cases, negative velocities of motion are considered equally admissible with positive velocities (arts. 2 and 6); and therefore when forces are spoken of as acting in the direction of motion, it is not meant to exclude the case of forces acting in a direction opposite to that in which the particle actually (or *positively*) moves.

From this law, again, another follows as a consequence, viz.:

(4) *The acceleration of motion produced by any number of forces, acting simultaneously on a particle in the direction of its motion, is equal to the sum of the accelerations which they would separately produce if each of them acted singly in the same direction.*

For if f, f', f'', &c. denote the statical effects of the forces, and a, a', a'', &c. their respective accelerating effects when they act singly; by the preceding law we have

$$\frac{\text{statical effect of the simultaneous action of the forces}}{\text{accelerating effect of their simultaneous action}}$$

$$= \frac{f}{a} = \frac{f'}{a'} = \frac{f''}{a''} = \&c.$$

$$\text{and} \therefore = \frac{f + f' + f'' + \&c.}{a + a' + a'' + \&c.}$$

But the statical effect of the forces acting simultaneously is $= f + f' + f'' + \&c.$; and therefore the acceleration produced by their simultaneous action is $= a + a' + a'' + \&c.$

(5) *The acceleration produced by a force acting in any direction on a given particle is in the direction and proportional to the intensity of the force;* or, in other words, it is precisely the same as would have been produced by the force had it acted on the particle either initially at rest or moving in the direction of the force.

This law contemplates the case of a force acting on a particle in a direction different from that in which it is moving; and it asserts that the motion of the particle is to be found by combining its velocity with an acceleration in the direction of the force and proportional to the force's intensity, according to the manner shewn in articles 8 and 9.

The correctness of the law may be tested by observing the motion of a heavy body thrown in a direction inclined to the vertical. The path of such a body is observed to be in the vertical plane passing through the line along which the body is initially projected: this is in accordance with the law, for there is no force acting on the body except in this plane. Let the straight line along which the body is initially projected be taken for the axis of x, and the vertical line drawn downwards through the point of projection for the axis of y. And let x, y be the co-ordinates of the body referred to these axes, at the end of the time t measured from the instant of projection. Also let v be the velocity with which the body is projected. Lastly, if the body were let fall vertically downwards so as to move freely under the action of its own weight, let g denote the acceleration of motion which it would experience. Since $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$

measure the component accelerations of motion of the body parallel respectively to the axes of x and y (art. 8), we must have, if the law be true,

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = g;$$

$$\therefore \quad \frac{dx}{dt} = C, \quad \frac{dy}{dt} = gt + C',$$

C, C' being quantities independent of x, y, and t. Now if v be the initial velocity of projection, we must have when $t = 0$, $\frac{dx}{dt} = v$, and $\frac{dy}{dt} = 0$, $\therefore$ $C = v$, and $C' = 0$;

$$\therefore \quad \frac{dx}{dt} = v, \quad \frac{dy}{dt} = gt.$$

Integrating a second time, we get

$$x = vt, \quad y = \tfrac{1}{2}gt^2,$$

no constants being added, since when $t = 0$, $x = 0$, and $y = 0$.

Eliminating t between these equations, we get for the equation of the path

$$x^2 = \frac{2v^2}{g}y,$$

the equation of a parabola which has its axis vertical and vertex upwards. The axis of y is clearly a diameter, and the axis of x is the tangent at the extremity of this diameter; also the distance of the point of projection from the focus or the directrix is $= \frac{v^2}{2g}$.

From these equations all the circumstances of the motion may be found, and the results may be compared with the motion actually observed. The resistance of the air to bodies moving through it prevents the perfect agreement of the results of the above process with the results of observation. But the agreement is sufficiently close to warrant the conclusion that the theory is correct.

Experiments on the motions of heavy bodies on inclined planes may also be made with the view of testing the truth

of the law. But the friction between the bodies and the planes will interfere, as well as the air's resistance, in experiments of this kind.

Thus far then, it is concluded, that if a force act on a material particle, it produces the same effect, whether the particle be at rest or in motion, and whatever be the direction in which the force acts; viz. an acceleration in the direction of the force proportional to the statical measure of the force. Hence any motion which a particle may have does not interfere so as to prevent a force acting on the particle from producing its full accelerating effect.

From this law the next follows as a consequence:

(6) *When any forces act simultaneously on a material particle, which is either at rest or in motion, each of them produces in its own direction an acceleration proportional to its intensity, and precisely the same as it would have produced if it alone had acted and the particle had been initially at rest.*

For the forces acting simultaneously produce a single force whose statical measure is the resultant of the statical measures of the forces; and this single force, by the previous law, produces an acceleration of motion in its direction proportional to its intensity. But the same result is obtained by supposing the forces to produce in their respective directions, accelerations proportional to their intensities, and then combining (by the principle of art. 8), these accelerations together so as to find their resultant acceleration.

Hence, in examining the motion of a particle, acted on simultaneously by any forces, it is a matter of indifference whether the accelerating effect of each force be considered and the motion inferred from the coexistence of all the accelerating effects, or whether the resultant of the forces be considered and its accelerating effect on the motion deduced. It will generally be most convenient to suppose the forces resolved into three (statically) equivalent forces acting in directions perpendicular to one another, and then to consider the accelerating effects of these forces in their respective directions.

Hitherto the actions of different forces on the same par-

ticle have been compared; and it has been concluded that a force whose statical measure is p produces in its direction an acceleration of motion a, such that $p \propto a$, and therefore

$$p = ma,$$

m being a quantity independent of p and a, and constant for the same particle. If, however, the actions of the same force on different particles be compared, the accelerations of motion produced will not necessarily be the same; therefore the quantity m is not necessarily the same for different particles. Consequently, each particle of matter has its peculiar and distinctive constant, which expresses the connexion between the statical measure of any force acting on it and the acceleration of its motion produced by the force; the greater this constant is, the greater will be the force required to produce a given acceleration of motion, and the less will be the acceleration of motion produced by a given force. The characteristic mechanical constant of a particle, thus found to exist, is taken as the measure of the *mass* of, or the *quantity of matter* in, the particle; it also may be considered as measuring the indisposition to a change of motion, or as it is called, the *inertia*, of the particle. For the sake of distinctness, the following definition may be made:

The mass of a particle is measured by the statical intensity of the force which produces in it an acceleration of motion equal to unity.

The accelerations of motion of heavy bodies, urged to the earth's surface by their weights only, are the same whatever be the weights of the bodies. The differences usually observed in the motions of falling bodies arise entirely from the resistance of the air, which acts differently on different bodies; as may be proved by letting bodies, whose falling motions are very different, such as a sovereign and a feather, fall together from the same height within a receiver out of which the air has been extracted, when they will be found to reach the bottom of the receiver at precisely the same instant. Heavy bodies whose weights are equal have therefore equal masses, and the masses of heavy bodies are proportional to their weights.

The *moving effect* of a force acting on a particle is measured by the product of the mass of the particle into the acceleration of motion which the force produces.

When a particle is in motion the product of its mass into its velocity at any instant is called its *momentum* at the instant; this product is sometimes called the *quantity of motion.* The product of the mass into the square of the velocity is called the *vis-viva* of the particle.

The laws of this article are summed up in the following:

Second Law of Motion. *If any material particles be acted on by external forces, the moving effect of each force is in the direction in which the force acts, and is proportional to the statical effect of the force.*

13. The foregoing laws relate to the motion of a single particle, the next law relates to the actions of particles on one another, and it is this:

Third Law of Motion. *If one particle act on another particle, the second exerts on the first a force equal in magnitude and opposite in direction to that which the first exerts on the second.*

In this law it is a matter of indifference whether the forces be measured by their moving effects, or by their statical effects; for by the second law these are equivalent. The actions of the particles on one another may be of any kind whatever, for instance, the forces which two particles exert on one another, when they are in contact, or when they are connected by a thread, or when they attract or repel one another. The law is often cited as the law of action and reaction, and is often stated thus:—There is always a reaction equal and opposite to action, or the actions of bodies are mutual, equal and opposite.

The accuracy of this law may be tested by observing the motion of two heavy bodies of different weights, connected by a thread, and hung over a pully. Let m and m' be the masses of the bodies; and let g be the accelerating effect of gravity, that is, the acceleration of motion of a heavy body which is allowed to fall freely under the action of its own

weight. Let the effects of the masses of the pully and the string on the motion be neglected.

The heavier body (m suppose) draws up the lighter by means of the connecting thread, and the lighter body prevents the heavier from moving so fast as it would if it were permitted to move freely. Let T be the moving effect of the force which m exerts on m'; if the law be true, m' exerts on m an equal and opposite force. Hence the moving effect of the force acting on m' to draw it up $= T - m'g$; and that of the force acting on m to draw it down $= mg - T$; the accelerating effects of these forces on the particles are respectively $\frac{T}{m'} - g$ and $g - \frac{T}{m}$. Now the thread being always stretched, the space through which m has fallen in any time is equal to that through which m' has risen in the same time; and therefore the acceleration of m's motion downwards is equal to the acceleration of m''s motion upwards;

$$\therefore \frac{T}{m'} - g = g - \frac{T}{m},$$

and, consequently, $= \frac{m - m'}{m + m'} . g$. This is the acceleration of motion of each of the bodies, the one upwards and the other downwards; since it is constant, the velocity of each body at the end of a time t measured from the beginning of motion

$$= \frac{m - m'}{m + m'} g\, t,$$

and the space described by each $= \frac{m - m'}{m + m'} . \frac{gt^2}{2}$ (art. 7, cor.)

These formulæ are found to express the circumstances of the motion actually observed, with considerable accuracy. Any discrepancies between the calculated and the observed motions are sufficiently accounted for by the mass of the pully having been left out of consideration, and by the resistance of the air, and the friction at the axis of the pully. An experiment of this kind may be considered as a test of the second and third laws taken together, or of either of

them if the other be considered as proved; for the above process is founded on the second law no less than on the third.

Experiments on the motion of two heavy bodies hung over a pully by a thread may be made very conveniently. For by making the weights of the bodies differ slightly, the motion can be rendered sufficiently slow to be easily and accurately observed. Besides, the effect of friction at the pully may be greatly diminished by making the axis rest on the circumferences of wheels; and the effects on the motion produced by the masses of the pully and the wheels may be taken account of. The resistance of the air also affects the motions of bodies which move slowly much less than those of bodies which move quickly.

CHAPTER III.

THE FREE MOTION OF A MATERIAL PARTICLE.

---

14. A PARTICLE is said to move *freely* under the action of forces, when these forces are the only causes which affect its motion; it is subject to no geometrical conditions, nor does it experience resistance from the medium in which it moves.

The forces which act on a particle may be supposed to be known at any instant when the accelerations which they produce in three perpendicular directions are known. At the end of the interval of time t measured from a fixed epoch, let x, y, z be the co-ordinates of a particle referred to a system of co-ordinate axes, and let X, Y, Z be the component accelerating effects, in the respective directions of the co-ordinate axes, of the forces acting on the particle. If the motion of the particle be free, we have

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y, \quad \frac{d^2z}{dt^2} = Z.$$

These are the equations of the free motion of a particle, and they may be regarded as the translation of the second law of motion into the language of analysis.

The equations of motion may be otherwise expressed. Let the forces acting on the particle be resolved in three perpendicular directions, one along the tangent to the particle's path in the direction of motion, one along the line of intersection of the normal plane with the osculating plane to the path in direction towards the concave side of the path, and the third perpendicular to the osculating plane. Let the accelerating effects of the forces in these directions be respectively denoted by S, N, and M. The second law of motion gives (by art. 8, cor. 1)

$$\frac{d^2s}{dt^2} = S, \quad \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 = N, \quad 0 = M,$$

s being the length of the path intercepted between a fixed point and the position of the particle at the end of the time t, and ρ the radius of absolute curvature of the path at the point where the particle is situated.

The forces may be resolved in any three perpendicular directions, and their accelerating effects estimated in these directions. If the position of a particle be determined by z its distance from the plane of xy, r its distance from the axis of z, and θ the angle between the plane zr and the plane zx; and if Z, P, Q be the accelerating effects of the forces in directions along z, along r, and perpendicular to zr, the equations of motion are (art. 8, cor. 2),

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = P, \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = Q, \quad \frac{d^2z}{dt^2} = Z.$$

Another set of equations may be obtained by estimating the accelerating effects of the forces in the directions specified in cor. 3 of art. 8.

All these sets of equations are equivalent to one another, and one set being given all the others may be derived from it. The only advantage of using one set rather than another is that troublesome eliminations may often be avoided in investigations of motions.

I. *The motion of a particle acted on by a force in the line of motion.*

15. When a particle is acted on by a force whose line of action always coincides with the direction of motion, it is clear that the path is a straight line. Suppose $A'OA$ to represent the straight line along which the particle moves. Let O be a fixed point in the line assumed as an origin from which to measure distances. At the end of the time t, measured from a fixed epoch, let P be the place of the particle. Let the distance OP be denoted by s; and let f be the accelerating effect of the force acting on the particle in the direction $A'PA$: the equation of motion of the particle then is

$$\frac{d^2s}{dt^2} = f.$$

In applying this equation to any particular case of motion, attention must be paid to the algebraical signs of s, f, and the velocity. Suppose that lines measured from O in the direction OA are considered positive: this supposition involves the following as necessary consequences:

(1) Lines measured from O in the direction OA' must be considered negative.

(2) Velocities of motion in the direction $A'OA$ must be considered positive, and velocities in the opposite direction AOA' must be considered negative. (see art. 2.)

(3) Accelerations of motion (or the accelerating effects of forces which act) in the direction $A'OA$ must be considered positive, and those in the opposite direction AOA' must be considered negative. (see art. 6.)

To make the system of algebraic signs complete, it may be added, that since positive periods of time refer to periods after a certain fixed epoch, negative periods of time refer to periods before the epoch. Negative times are seldom considered, because questions of motion generally are of the form—What will the circumstances of a particular motion be? and not—What have the circumstances of the motion been, or what would they have been if the law of the particular motion had been always the same? The only cases in which negative periods of time will enter will be as negative roots of equations, formed for determining the instants at which certain given circumstances happen.

16. *To determine the motion of a particle acted on by a constant force in the line of motion.*

At the end of the time t measured from a fixed epoch, let s be the distance of the particle from a fixed point in its path, v its velocity, and f the accelerating effect of the force acting on it, which is constant. At the beginning of the time t, let a be the distance of the particle from the fixed point, and v' the velocity. We have

$$\frac{d^2s}{dt^2}=f, \quad \frac{ds}{dt}=v.$$

By integrating the first of these relative to t,

$$\frac{ds}{dt} = ft + C,$$

C being independent of s and t. Now when $t = 0$, $\frac{ds}{dt}$ or v is v'; $\therefore C = v'$, and hence

$$\frac{ds}{dt} = v' + ft = v,$$

which gives the velocity at any instant of the motion.

We might find the velocity of the particle at any point of its path from the same equation; for from this equation

$$2\frac{ds}{dt}\cdot\frac{d^2s}{dt^2} = 2f\frac{ds}{dt};$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = 2fs + C',$$

and when $s = a$, $\frac{ds}{dt} = v'$; $\therefore v'^2 = 2fa + C'$, and hence

$$\left(\frac{ds}{dt}\right)^2 = v'^2 + 2f(s - a);$$

$$\therefore \frac{ds}{dt}, \text{ or } v = \sqrt{v'^2 + 2f(s - a)}.$$

The sign of the radical will be noticed presently.

Integrating the former equation for the velocity, we get

$$s = v't + \tfrac{1}{2}ft^2 + C''$$

when $t = 0$, $s = a$; $\therefore a = C''$, and consequently

$$s = a + v't + \tfrac{1}{2}ft^2,$$

which gives the position of the particle at any instant.

The expression for the particle's velocity at a point of its path has two values equal in magnitude but opposite in sign. This shews that the particle passes a point at distance s from the origin twice (unless s have such a value as makes $v'^2 + 2f(s - a)$ zero or negative). The times when the par-

ticle passes the point are to be found from the equation which expresses s in terms of t, and which may be written

$$t^2 + \frac{2v'}{f}t - \frac{2(s-a)}{f} = 0.$$

The roots of this equation are

$$\frac{\sqrt{v'^2 + 2f(s-a)} - v'}{f}, \text{ and } -\frac{\sqrt{v'^2 + 2f(s-a)} + v'}{f},$$

in both of which the radical is supposed to be taken with the positive sign. In order therefore to explain fully the motion, it is necessary to consider negative times. In fact the question which has been investigated is this:—To determine the motion of a particle which moves continually in a straight line under the action of a constant force in the line of motion, subject only to the condition that at a given instant it shall be at a distance a from the origin of distances, and moving with a velocity v'. The solution is given by the equations

$$v = v' + ft, \quad s = a + v't + \tfrac{1}{2}ft^2,$$

in which negative periods of time are really no more excluded than positive periods; or, in other words, these are the equations of motion of the particle as well before as after the given instant. To consider then the motion. When $t = -\infty$, $s = +\infty$, and $v = -\infty$, that is, at an infinitely great time before the instant which is the origin of times, the particle is on the positive side of the origin at an infinite distance from it, and is moving with an infinite velocity from the positive to the negative side. The particle continues to move in the same direction, but with a continually diminishing rate of motion, till the time $t = -\frac{v'}{f}$, when it comes to rest at a distance from the origin $s = a - \frac{v'^2}{2f}$ (this point being on the positive or negative side of the origin according as a is greater or less than $\frac{v'^2}{2f}$). Immediately after this the particle begins to move in the opposite direction, viz. from negative to positive, with a continually increasing velocity, and

passes successively, but in a reverse order, all the points of the path which it formerly passed while moving from positive to negative; and its velocity at any point is the same as in the former passage at the same point. The particle then continues to move for ever in this direction with a continually increasing velocity. Every point of the line of motion, therefore, which lies at a greater distance from the origin than $a - \frac{v'^2}{2f}$, is passed twice by the particle, once when the particle is moving from positive to negative, which corresponds to the second root above of the equation in t, and again, when the particle is moving from negative to positive, which corresponds to the other root. Thus the form of the expression for the velocity $v = \sqrt{v'^2 + 2f(s-a)}$ is completely explained.

Although all algebraical signs and roots of equations which enter into investigations concerning the motions of particles are perfectly relevant to the motions, and indeed are necessary to be considered in order completely to determine the motions; yet often it is sufficient to consider particular signs and roots only, viz. such only as refer to particular parts of the motions which are more immediately the objects of inquiry. Thus generally it will be sufficient to take notice of positive roots only of equations for determining the times when particular events take place, because generally motions are supposed to begin at the instant from which times are measured. And in the same way other limitations as to algebraical signs and roots of equations are frequently introduced; but with regard to such limitations no general rule can be given; the proper course to be followed is generally so manifest that no rule is required.

Cor. 1. In the foregoing investigation the quantities a, v', and f have been considered positive; in other words, the particle has been supposed to be initially at a distance from the origin on the positive side equal to a, and to be moving with a velocity v' from the negative to the positive side, and the force has been supposed to act in the direction from the negative to the positive side with an accelerating intensity

equal to f. But the symbols a, v', and f may be regarded not simply as numerical symbols but as algebraical ones, that is, they may be considered as including the signs as well as the magnitudes of the quantities which they denote; thus the preceding investigation will be made to include every possible particular case of the kind of motion investigated. Suppose, for instance, a particle to be initially on the negative side of the origin at a distance from it expressed by the numerical quantity A, and to be moving with the numerical velocity V in the direction from the positive to the negative side; suppose it to be acted on by a force whose numerical accelerating effect is F in the direction from positive to negative: if in the preceding equations we put

$$a = -A, \quad v' = -V, \quad f = -F,$$

we get

$$v = -V - Ft, \quad s = -A - Vt - \tfrac{1}{2}Ft^2;$$

which equations determine fully the motion.

Hence if the symbols employed in any investigation be understood in this extended sense, the investigation may be regarded as including in it all particular cases of the same kind of motion as that which is investigated.

COR. 2. The force by which heavy bodies are urged towards the earth's surface (or the force of gravity as it is called) being a constant force for small distances above the surface, the preceding investigation includes in it the whole theory of falling bodies.

It is found that heavy bodies are urged downwards with an acceleration of 32.2 feet in a second; this number is generally denoted by the letter g.

Let a heavy particle falling from rest describe the space s in t seconds, and let v be its velocity at the end of the same time. We have the following formulæ:

$$v = gt = \sqrt{2gs},$$

$$s = \tfrac{1}{2}gt^2 = \tfrac{1}{2}vt = \frac{v^2}{2g},$$

$$t = \frac{v}{g} = \frac{2s}{v} = \sqrt{\frac{2s}{g}};$$

from which every circumstance connected with the motion may be found.

If a heavy particle be projected downwards with a velocity v', and if in t seconds afterwards v be its velocity and s the space through which it has fallen; the formulæ of its motion are

$$v = v' + gt = \sqrt{v'^2 + 2gs},$$

$$s = v't + \tfrac{1}{2}gt^2 = \frac{v + v'}{2}\,t = \frac{v^2 - v'^2}{2g},$$

$$t = \frac{v - v'}{g} = \frac{2s}{v + v'} = \frac{\sqrt{v'^2 + 2gs} - v'}{g}.$$

Again, if a heavy particle be projected upwards with a velocity v' and if in t seconds afterwards v be its upward velocity, and s the space through which it has risen; the formulæ of motion are

$$v = v' - gt = \sqrt{v'^2 - 2gs},$$

$$s = v't - \tfrac{1}{2}gt^2 = \frac{v' + v}{2}\,t = \frac{v'^2 - v^2}{2g},$$

$$t = \frac{v' - v}{g} = \frac{2s}{v' + v} = \frac{\sqrt{v'^2 - 2gs} + v'}{g}.$$

From these formulæ every question about falling bodies may be solved.

For example. Let a heavy particle be projected upwards with a velocity v'. At the end of the time t its upward velocity is $v' - gt$, and therefore at the time $\frac{v'}{g}$ it is brought to rest; afterwards the upward velocity becomes negative, that is, the particle begins to move downwards. Again, the distance of the particle above the point of projection at the end of the time t is $v't - \frac{1}{2}gt^2$, and therefore the greatest height to which it rises is $v'.\frac{v'}{g} - \frac{1}{2}g.\frac{v'^2}{g^2} = \frac{v'^2}{2g}$; also it is at the point of projection at the times given by the equation $v't - \frac{1}{2}gt^2 = 0$, that is, when $t = 0$, the instant of projection,

and when $t = \frac{2v'}{g}$, the instant of the particle's passing the point of projection in its downward motion.

17. *A particle is placed at a given distance from a fixed point, towards which it is attracted by a force whose intensity at any instant is proportional to the distance of the particle from the point; to determine the subsequent motion.*

The path of the particle will manifestly be a straight line drawn through the point towards which the force tends and through the initial position of the particle. Let the fixed point be taken for the origin of distances, and let a be the initial distance of the particle. After the time t measured from the beginning of the motion, let s be the distance of the particle from the point. The accelerating effect of the force acting on the particle may be represented by μs in direction from the particle to the point, μ being the accelerating effect of the force when the particle's distance from the point is unity. Hence the equation of motion is

$$\frac{d^2s}{dt^2} = -\mu s.$$

Multiplying each side of this equation by $2\frac{ds}{dt}$, and integrating, we have

$$\left(\frac{ds}{dt}\right)^2 = -\mu s^2 + C.$$

Now when $s = a$, $\frac{ds}{dt} = 0$, hence, $0 = -\mu a^2 + C$;

$$\text{and } \therefore \left(\frac{ds}{dt}\right)^2 = \mu (a^2 - s^2);$$

$$\text{and } \frac{ds}{\sqrt{a^2 - s^2}} = \sqrt{\mu} \,.\, dt.$$

Of course both signs of the radical are admissible. The negative sign will be taken, because immediately after the

beginning of the motion s decreases as t increases; hence by integrating,

$$\cos^{-1}\left(\frac{s}{a}\right) = \sqrt{\mu}\,t + C'$$

when $t = 0$, $s = a$, and $\therefore\ C' = \cos^{-1}(1) = 0$, or 2π or &c.

Hence, $s = a \cos \sqrt{\mu}\,t.$

The same result is arrived at by taking the positive sign of the radical above. If v be the velocity of the particle,

$$v = -a\sqrt{\mu}\sin\sqrt{\mu}\,t.$$

These equations give the position and velocity of the particle at any instant. They shew that the particle begins to move from + to −; that at the time $t = \frac{\pi}{2\sqrt{\mu}}$ it reaches the origin, and has its greatest possible velocity $a\sqrt{\mu}$ in direction from + to −; that it afterwards continues to move in the same direction with a diminishing velocity, reaches its greatest distance from the origin on the negative side at the time $t = \frac{\pi}{\sqrt{\mu}}$, and is then reduced to rest; that, immediately after, it begins to move in the opposite direction from − to +, reaches the origin for the second time at the time $t = \frac{3\pi}{2\sqrt{\mu}}$, and has again its greatest velocity; that it passes the origin, moves on the positive side with a diminishing velocity, and at the time $t = \frac{2\pi}{\sqrt{\mu}}$ reaches the point where it initially was, and is there again reduced to rest. It then repeats precisely the same motion again and again.

The particle therefore oscillates continually between two points equidistant from, and on opposite sides of, the origin; and the time occupied in passing from one to the other of these points, or the time of an oscillation, is $= \frac{\pi}{\sqrt{\mu}}$. It is re-

markable that this time is independent of the distance between the extreme points of vibration.

The time in which the particle moves from its initial position to the origin is $=\frac{\pi}{2\sqrt{\mu}}$, which is independent of a the initial distance between the particle and the origin. Hence the particle will reach the origin in the same time from whatever point it begin initially to move.

18. *A particle attracted towards a fixed point reaches the point in the same time from wherever its motion begins: to find the law of the attractive force.*

Let the fixed point, towards which the particle is attracted, be taken for the origin of distances, and let a be the initial distance of the particle from the origin. After the time t elapsed from the beginning of the motion, let s be the distance of the particle from the origin, and $f(s)$ the accelerating effect of the attractive force acting on it at that distance,

$$\frac{d^2 s}{dt^2} = -f(s),$$

$$\text{and } \therefore \left(\frac{ds}{dt}\right)^2 = C - 2\int f(s)\, ds$$

$$= C - \phi(s) \text{ suppose,}$$

where $\phi(s)$ is such that $\frac{d}{ds}\phi(s) = 2f(s)$.

Now when $s = a$, $\frac{ds}{dt} = 0$; $\therefore 0 = C - \phi(a)$, and we have

$$\left(\frac{ds}{dt}\right)^2 = \phi(a) - \phi(s);$$

$$\therefore dt = -\frac{ds}{\sqrt{\phi(a) - \phi(s)}},$$

the negative sign of the radical being taken since s decreases as the time increases. Hence integrating, and determining the constant so that t may $= 0$ when $s = a$,

$$t = \int_s^a \frac{ds}{\sqrt{\phi(a) - \phi(s)}};$$

which gives the length of time between the beginning of the motion and the instant when the particle is at distance s from the origin. If s be put $= 0$, we shall get the time in which the particle reaches the origin; calling T this time

$$T = \int_0^a \frac{ds}{\sqrt{\phi(a) - \phi(s)}}$$

$$= \int_0^a \frac{ds}{\sqrt{\phi(a)}} \left[1 + \tfrac{1}{2} \cdot \frac{\phi(s)}{\phi(a)} + \frac{1.3}{2.4} \cdot \left\{\frac{\phi(s)}{\phi(a)}\right\}^2 + \dots\dots \right.$$

$$\left. + \frac{1.3.5\dots(2i-1)}{2.4.6\dots 2i} \cdot \left\{\frac{\phi(s)}{\phi(a)}\right\}^i + \dots \right]$$

$$= \frac{a}{\{\phi(a)\}^{\frac{1}{2}}} + \tfrac{1}{2} \cdot \frac{\int_0^a \phi(s)\,ds}{\{\phi(a)\}^{\frac{3}{2}}} + \dots + \frac{1.3.5\dots(2i-1)}{2.4.6\dots 2i} \frac{\int_0^a \{\phi(s)\}^i ds}{\{\phi(a)\}^{i+\frac{1}{2}}} + \dots$$

And this by supposition is independent of a; which can only be by having

$$\phi(a) = k \,.\, a^2,$$

in which k is a constant;

$$\therefore\ T = \frac{1}{k^{\frac{1}{2}}} \left(1 + \frac{1}{2} \cdot \frac{1}{3} + \frac{1.3}{2.4} \cdot \frac{1}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{1}{7} + \dots\dots\right)$$

$$= \frac{\pi}{2k^{\frac{1}{2}}}.$$

Hence $f(s) = \frac{1}{2} \frac{d}{ds} \cdot \phi(s) = \frac{1}{2} \frac{d}{ds}(ks^2) = ks$; that is, the force is proportional to the distance of the particle from the origin.

19. *To determine the motion of a particle attracted towards a fixed point by a force whose intensity at any instant is inversely proportional to the square of the distance between the particle and the point; the particle being initially at rest at a given distance from the point.*

The motion will be wholly in the straight line drawn through the fixed point and the initial position of the particle. Let the fixed point be taken for the origin of distances; let

a be the initial distance of the particle from it; and let s be the distance of the particle at the end of the time t measured from the beginning of the motion. The accelerating effect of the force may be represented by $\frac{\mu}{s^2}$, in which μ is the accelerating effect at the unit of distance. Hence $-\frac{\mu}{s^2}$ will represent the force when the particle is on the positive side of the origin, and $+\frac{\mu}{s^2}$ will be the force when the particle is on the negative side. Therefore the equation of motion is

$$\frac{d^2 s}{dt^2} + \frac{k\mu}{s^2} = 0,$$

where k is a discontinuous quantity which $= +1$ when s is positive, and $= -1$ when s is negative. By integrating this equation we have

$$\left(\frac{ds}{dt}\right)^2 = \frac{2k\mu}{s} + C.$$

Now when $s = a$, $\frac{ds}{dt} = 0$; $\therefore\ 0 = \frac{2\mu}{a} + C$, and therefore

$$\left(\frac{ds}{dt}\right)^2 = 2\mu\left(\frac{k}{s} - \frac{1}{a}\right)$$

$$= 2\mu\left(\frac{1}{s} - \frac{1}{a}\right) \text{ for positive values of } s$$

$$= 2\mu\left(-\frac{1}{s} - \frac{1}{a}\right) \text{ for negative values of } s.$$

It becomes necessary here to examine the motion of the particle first on the positive side of the origin, and then on the negative. When it moves on the positive side we have

$$\frac{ds}{dt} = -\sqrt{\frac{2\mu(a-s)}{as}},$$

the negative sign of the radical being taken, because after the motion begins s decreases while t increases. Hence

$$\sqrt{\frac{2\mu}{a}}\,.\,t = -\int \frac{s\,ds}{\sqrt{as - s^2}}$$

$$= -\int \frac{\left(s - \frac{a}{2}\right) ds}{\sqrt{\frac{a^2}{4} - \left(s - \frac{a}{2}\right)^2}} - \frac{a}{2}\int \frac{ds}{\sqrt{\frac{a^2}{4} - \left(s - \frac{a}{2}\right)^2}}$$

$$= \sqrt{\frac{a^2}{4} - \left(s - \frac{a}{2}\right)^2} + \frac{a}{2}\cos^{-1}\left(\frac{s - \frac{a}{2}}{\frac{a}{2}}\right) + C'.$$

When $t = 0$, $s = a$, therefore $0 = C'$, and therefore

$$t = \sqrt{\frac{a}{2\mu}}\left(\sqrt{as - s^2} + \frac{a}{2}\cos^{-1}\frac{2s - a}{a}\right).$$

If in this s be put $= 0$, we get the time of the particle's reaching the point to which it is attracted

$$= \frac{\pi}{2}\sqrt{\frac{a^3}{2\mu}},$$

and at the point it is moving with an infinite velocity from positive to negative. It becomes now necessary, therefore, to consider the motion on the negative side; for this purpose put $-s'$ for s in the equation which expresses the velocity on the negative side of the origin, and we have

$$\frac{ds'}{dt} = +\sqrt{\frac{2\mu\,(a - s')}{as'}},$$

the positive sign of the radical being taken, because immediately after the particle passes the origin s' increases with the time.

Integrating this equation as before, there results

$$\sqrt{\frac{2\mu}{a}}\,t = C'' - \sqrt{as' - s'^2} - \frac{a}{2}\cos^{-1}\frac{2s' - a}{a}.$$

And when $t = \frac{\pi}{2}\sqrt{\frac{a^3}{2\mu}}$, $s' = 0$; $\therefore\ \frac{\pi a}{2} = C'' - \frac{\pi a}{2}$;

$$\therefore\ t = \sqrt{\frac{a}{2\mu}}\left(\pi a - \sqrt{as' - s'^2} - \frac{a}{2}\cos^{-1}\frac{2s' - a}{a}\right).$$

This gives the motion of the particle from the origin on the negative side; it shews that at the time $t = \pi\sqrt{\frac{a^3}{2\mu}}$, $s' = a$, and the particle is then reduced to rest. After having reached this its greatest distance from the origin on the negative side, the particle begins to move back towards the origin, and the equations relating to this part of the motion are

$$\frac{ds'}{dt} \text{ or } - (\text{the velocity}) = -\sqrt{\frac{2\mu(a-s')}{as'}},$$

$$t = \sqrt{\frac{a}{2\mu}}\left(\pi a + \sqrt{as' - s'^2} + \frac{a}{2}\cos^{-1}\frac{2s'-a}{a}\right),$$

as may be easily found. Hence the particle arrives at the origin a second time at the time $t = \frac{3\pi}{2}\sqrt{\frac{a^3}{2\mu}}$, and is then moving with an infinite velocity from the negative to the positive side of the origin. When the particle passes the origin and is moving on the positive side from the origin, the equations become

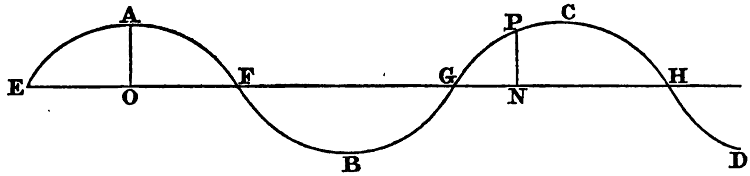
$$\text{velocity} = +\sqrt{\frac{2\mu(a-s)}{as}},$$

$$t = \sqrt{\frac{a}{2\mu}}\left(2\pi a - \sqrt{as - s^2} - \frac{a}{2}\cos^{-1}\frac{2s-a}{a}\right),$$

and the particle reaches the point from which its motion began at the time $t = 2\pi\sqrt{\frac{a^3}{2\mu}}$, and is there reduced to rest. It is now in precisely the same circumstances as at the time $t = 0$; and it proceeds to repeat the same motion.

The particle thus oscillates continually between two points equidistant from the origin and on opposite sides of it; and the time of an oscillation is $= \pi\sqrt{\frac{a^3}{2\mu}}$.

The following geometrical construction follows from what goes before. Let AO be drawn equal to a, and on it, as



an axis, construct the cycloid EAF. Produce the base EOF indefinitely; lay the cycloids FBG, GCH, &c., which are all equal to EAF, with their bases in the line $EFGH$, joining on to one another, and having their vertices alternately on one side and the other of $EFGH$, thus forming a continuous curve line $EAFBGCHD$. If, now, t be any time from the commencement of motion, and ON be made $= t\sqrt{\dfrac{2\mu}{a}}$; the ordinate PN, drawn perpendicular to $EFGH$, will represent the distance of the particle from the point to which it is attracted, and the particle will be on one side or other of the point according as PN is on one side or other of the line $EFGH$.

20. If a particle be attracted towards a fixed point, and be initially either at rest or moving to or from the point, its path will be a straight line passing through its initial position and the point to which it is attracted. Let the point be taken for the origin from which to measure distances, and let s be the distance of the particle from the origin at the end of the time t elapsed since a fixed epoch. Suppose the force of attraction at any instant to be proportional to the n^{th} power of the distance of the particle from the origin at the instant, and suppose μ to be the accelerating effect of the attraction on the particle when its distance from the origin is unity. In order to determine the motion, we must substitute for f, in the equation

$$\frac{d^2s}{dt^2} = f,$$

the quantity μs^n with such a sign that it may be negative for all positive values of s, and positive for all negative values of s. Hence

(1) If n be of the form $\frac{2i+1}{2i'+1}$ in which i, i' are any whole numbers positive or negative (not excluding zero), the equation of motion is

$$\frac{d^2 s}{dt^2} + \mu s^n = 0.$$

(2) If, again, n be of the form $\frac{2i}{2i'+1}$, the equation of motion is

$$\frac{d^2 s}{dt^2} + k\mu s^n = 0,$$

where k is a quantity that $= +1$ for all positive values of s, and $= -1$ for all negative values of s.

(3) If, lastly, n be of the form $\frac{2i+1}{2i'}$, the equation is

$$\frac{d^2 s}{dt^2} + k\mu (ks)^n = 0.$$

The first two equations are manifestly included in the third; hence the general equation is

$$\frac{d^2 s}{dt^2} + \mu k^{n+1} s^n = 0,$$

the symbolical expression $\frac{1 - 0^s}{1 + 0^s}$ may be put instead of k.

Unless n be of the form $\frac{2i+1}{2i'+1}$, the investigation of the motion will generally consist of two parts referring to the motions of the particle on the positive and negative sides of the origin; for it will generally be impossible to find a single integral of the equation of motion which includes in it the different values of k. Even sometimes when n is of the form $\frac{2i+1}{2i'+1}$ it may be necessary to consider the motions on the positive and negative sides of the origin apart; on account of the form of the integral being not sufficiently general to include both. For example, let $i = 0$, and $i' = -1$; the equation of motion is

$$\frac{d^2s}{dt^2} = -\frac{\mu}{s},$$

which expresses the motion of a particle attracted towards the origin by a force whose intensity is inversely proportional to the distance of the particle from the origin. Now the integral of this equation is

$$\left(\frac{ds}{dt}\right)^2 = C - 2\mu \log s$$

if the particle be on the positive side of the origin; but if the particle be on the negative side the integral is

$$\left(\frac{ds}{dt}\right)^2 = C - 2\mu \log(-s).$$

Both however may be included in one, by using the quantity k,

$$\left(\frac{ds}{dt}\right)^2 = C - 2\mu \log(ks).$$

Suppose the motion to begin when the particle is at distance a from the origin; $\therefore 0 = C - 2\mu \log a$, and

$$\left(\frac{ds}{dt}\right)^2 = 2\mu \log\left(\frac{a}{ks}\right).$$

This equation cannot be integrated further. But the time of reaching the origin may be found; it

$$= \frac{1}{\sqrt{2\mu}}\int_0^a \frac{ds}{\sqrt{\log\left(\frac{a}{s}\right)}} = \frac{2a}{\sqrt{2\mu}}\int_0^\infty \epsilon^{-x^2}dx$$

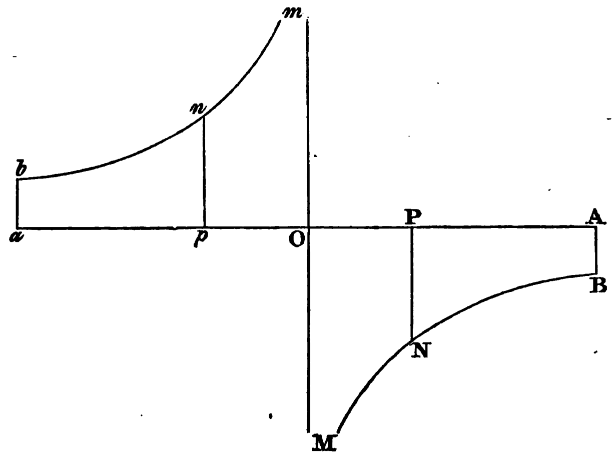
$\left(\text{found by putting } x^2 = \log\frac{a}{s}\right)$

$$= a\sqrt{\frac{\pi}{2\mu}}.$$

The particle manifestly oscillates between two points equidistant from the origin on opposite sides of it.

Although, in order to trace fully the motion of a particle by means of its equation of motion, the above circumstances must be attended to; yet, since the motions on opposite

sides of the origin must necessarily be precisely similar, it is sufficient to trace the motion on one side only of the origin. This may be illustrated in a geometrical way:—Let O be



the point to which the particle is attracted, and let AOa be the line of motion. Also let $BMmb$ be a curve such that the ordinate drawn from any point in Aa represents the accelerating effect of the force on the particle when it is at that point; since the particle is attracted towards O equally at equal distances, the branch bm will be precisely similar to the branch BM, and they will lie on opposite sides of Aa.

Let A be the point from which the particle begins to move, and let P be the position of the particle at a certain time afterwards; and draw the ordinates PN, AB. The area $ABNP$ represents half the square of the velocity of the particle at P in direction AOa. When the particle reaches O it is moving in direction Oa with a velocity the half of whose square is represented by the area $ABMO$; but at this point there is no force acting on the particle, because the direction of the force is neither from positive to negative nor from negative to positive, and therefore the particle proceeds to move on the negative side of the origin.

Let p be a point on the negative side of the origin such that $Op = OP$. When the particle is at p, half the square of its velocity is represented by the (algebraical) sum of the

areas $ABMO$, $Omnp$, which is equivalent to the area $ABNP$. When the particle reaches a, its velocity is zero. It then moves back again towards the origin, and when it comes to p in its backward motion, half the square of its velocity is represented by the area $apnb$; and it continues to oscillate between A and a.

Such is the way of representing geometrically the first integration of the equation of motion. (NEWTON'S *Principia*, Book I. Prop. 39.)

II. *The motion of a particle acted on by a force which is always parallel to a fixed straight line.*

21. If a particle be acted on by a force whose direction is always parallel to a fixed straight line, and be initially projected in a direction which is not parallel to the straight line, its path will be a curve wholly in one plane, viz. the plane drawn parallel to the fixed line through the line of initial projection. The position of the particle at any instant may therefore be referred to co-ordinate axes in the plane of motion. Let a straight line drawn in this plane perpendicular to the line of action of the force be taken for the axis of x, and another line in the same plane parallel to the line of action of the force for the axis of y. At the end of the time t measured from a fixed instant, let x, y be the co-ordinates of the particle, and let f be the accelerating effect of the force acting on it in direction parallel to the axis of y. The equations of motion are

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = f,$$

from which everything connected with the motion is to be found.

From the first of these equations it results that $\frac{dx}{dt}$ is equal to a constant quantity; that is, the component velocity of the particle in the direction perpendicular to the line of the force's action is the same at every instant during the motion. This may be regarded as the characteristic property of the kind of motion in question.

If a denote the constant velocity of the particle parallel to the axis of x, then $\frac{dx}{dt} = a$, and

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} = a\frac{dy}{dx};$$

$$\therefore\ \frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2}.$$

And substituting this in the second equation of motion, we get

$$a^2\frac{d^2y}{dx^2} - f = 0,$$

which is the differential equation of the path, when the force is given in terms of the position of the particle. From this also may be found the force required for the description of any given path.

22. *A particle describes a parabola under the action of a force parallel to the axis; to find the accelerating effect of the force.*

Let the vertex of the parabola be taken for origin, and the axis for the axis of y. Its equation is therefore

$$x^2 = 4my,$$

$4m$ being the latus-rectum.

Let x, y be the co-ordinates of the particle at the end of the time t, and f the accelerating effect of the force at the end of the same time. We have

$$\frac{d^2x}{dt^2} = 0,\quad \frac{d^2y}{dt^2} = f.$$

From the first of these equations $\frac{dx}{dt} = a$, an invariable component velocity of the particle perpendicular to the direction of the force. Also, from the equation of the parabola,

$$\frac{dy}{dt} = \frac{x}{2m}\cdot\frac{dx}{dt} = \frac{ax}{2m},$$

and, again differentiating

$$\frac{d^2y}{dt^2} = \frac{a}{2m}\frac{dx}{dt} = \frac{a^2}{2m};$$

$$\therefore f = \frac{a^2}{2m}.$$

Which shews that the force is constant. It is clear that a is the velocity of the particle at the vertex of the parabola; hence the accelerating effect of the force is

$$= \frac{(\text{velocity at the vertex})^2}{\text{half the latus-rectum}}.$$

23. *A particle decribes a conic section under the action of a force perpendicular to one of its axes; to find the force at any point of the path.*

If the axis of the conic perpendicular to which the force acts be taken for the axis of x, the equation of the path may be written

$$y^2 = 2mx - nx^2;$$

$2m$ being the latus-rectum, and n a constant which is positive in the ellipse, zero in the parabola, and negative in the hyperbola. By differentiation we have

$$y\frac{dy}{dx} = m - nx;$$

$$\therefore y\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)^2 - n = -\frac{m^2}{y^2}.$$

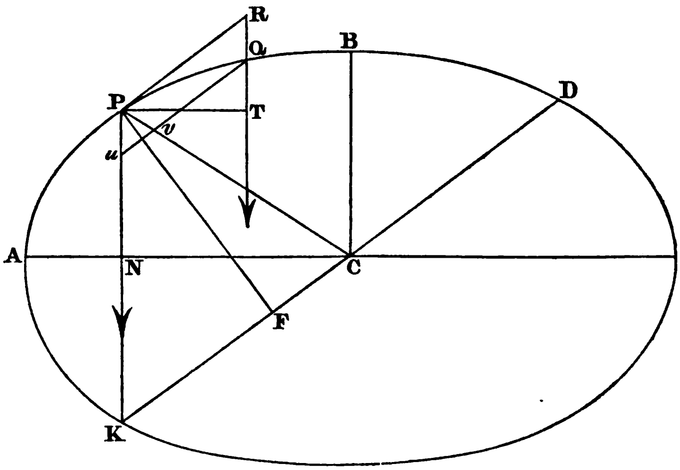
Hence if a be the constant component velocity of the particle perpendicular to the direction of the force, the accelerating effect of the force in the (positive) direction of the axis of y is

$$= a^2\frac{d^2y}{dx^2} = -\frac{a^2m^2}{y^3}.$$

The force therefore tends to the axis of the curve, and from point to point of the path it varies inversely as the cube of the particle's perpendicular distance from the axis.

The same result may be arrived at in the following way:

Suppose the path to be an ellipse *APD*, and suppose the



force to act perpendicular to the axis *AC*. Let the particle describe the arc *PQ* in the small time δt. Draw the tangent *PR*; *PN* and *QR* perpendicular to *AC*; *Qu* and *CD* parallel to *PR*; and *PT*, *PF* perpendicular respectively to *RQ* and *CD*.

Since the path deflects from *PR* towards *AC*, therefore the force acts in direction *PN* or *RQ*; that is, always towards *AC*.

If a be the velocity parallel to *AC*, and f be the accelerating effect of the force at *P*, then $PT = a \cdot \delta t$, and *RQ* is ultimately $= \frac{1}{2} f \delta t^2$ when δt is indefinitely diminished. Hence

$$f = 2 \text{ limit of } \frac{RQ}{\delta t^2} = 2 \text{ limit of } \frac{Pu}{PT^2}\left(\frac{PT}{\delta t}\right)^2,$$

$$= 2a^2 \text{ limit of } \frac{Pu}{PT^2}$$

$$= 2a^2 \text{ limit of } \frac{Pu}{Pv} \cdot \frac{Pv}{Qv^2} \cdot \frac{PR^2}{PT^2}, \text{ since } PR \text{ is ultimately} = Qv.$$

$$\text{Now } \frac{Pu}{Pv} = \frac{PK}{PC}, \quad \frac{Qv^2}{Pv} = \frac{CD^2}{CP^2}(CP + Cv), \text{ and } \frac{PR}{PT} = \frac{PK}{PF};$$

$$\therefore\ f = 2a^2 \text{ limit of } \frac{PK}{PC} \cdot \frac{CP^2}{CD^2\,(CP + Cv)} \cdot \frac{PK^2}{PF^2}$$

$$= \frac{a^2 \,.\, PK^3}{CD^2 \,.\, PF^2}.$$

But $PK \,.\, PN = BC^2$, and $CD \,.\, PF = AC \,.\, BC$;

$$\therefore\ f = \frac{a^2 \,.\, BC^4}{PN^3 \,.\, AC^2}, \text{ the same as before.}$$

24. *To determine the motion of a heavy particle, which is projected in a direction inclined to the vertical; the motion being supposed to take place in a perfect vacuum.*

The path will be wholly in the vertical plane passing through the line of initial projection. Let the point of projection be taken for origin, the horizontal line through the point of projection in the plane of motion for the axis of x, and the line drawn vertically upwards from the point of projection for the axis of y; and let x, y be the co-ordinates of the particle at the end of the time t, measured from the instant of projection. Suppose the particle projected with the velocity v' in a direction making an angle i with the horizon.

The equations of motion are

$$\frac{d^2x}{dt^2} = 0,\quad \frac{d^2y}{dt^2} = -g;$$

g being the accelerating effect of gravity. Hence

$$\frac{dx}{dt} = C,\quad \frac{dy}{dt} = C' - gt.$$

Now when $t = 0$, $\frac{dx}{dt} = v' \cos i$ and $\frac{dy}{dt} = v' \,.\, \sin i$;

$$\therefore\ \frac{dx}{dt} = v' \cos i,\quad \frac{dy}{dt} = v' \sin i - gt.$$

Integrating again and making x and y both vanish with t, we get

$$x = v't \cos i,\quad y = v't \sin i - \tfrac{1}{2}gt^2,$$

which give the position of the particle at any time after the instant of projection.

Eliminating t between these equations, we get for the equation of the path

$$y = x \tan i - \frac{gx^2 \sec^2 i}{2v'^2},$$

$$\text{or } \frac{2v'^2 \cos^2 i}{g}\left(\frac{v'^2 \sin^2 i}{2g} - y\right) = \left(x - \frac{v'^2 \sin 2i}{2g}\right)^2,$$

a parabola whose axis is vertical, and vertex upwards; its latus-rectum is $= \frac{2v'^2}{g}\cos^2 i$; and the co-ordinates of its vertex are $x = \frac{v'^2}{2g}\sin 2i$, $y = \frac{v'^2}{2g}\sin^2 i$.

Hence the greatest height to which the particle rises above the horizontal plane through the point of projection is

$$= \frac{v'^2}{2g}\sin^2 i.$$

The distance of the point of projection below the directrix is (equal to the greatest height + $\frac{1}{4}$th of the latus rectum)

$$= \frac{v'^2}{2g}.$$

If v be the velocity at the point (x, y) of the path

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = v'^2 - 2gv't\sin i + g^2t^2$$

$$= v'^2 - 2gy = 2g\left(\frac{v'^2}{2g} - y\right);$$

and therefore the velocity at any point of the path is the same as would have been acquired by a heavy particle falling freely to the point from the directrix.

The times when the particle is in the horizontal plane through the point of projection will be found by putting $y = 0$; they are therefore given by the equation

$$v't\sin i - \tfrac{1}{2}gt^2 = 0.$$

$t = 0$ gives the instant of projection, and $t = \frac{2v'\sin i}{g}$ is the time between the instant of projection, and the instant when

the particle is again in the horizontal plane through the point of projection. The distance between the point of projection and the point in which the particle meets this plane, or the horizontal range as it is called, will be found by putting, in the equation expressing x in terms of t, $\frac{2v' \sin i}{g}$ for t, and it is therefore

$$= \frac{v'^2 \sin 2i}{g}.$$

If through the point of projection a plane be drawn perpendicular to the plane of motion, and inclined to the horizon at angle α; and if the particle meet this plane at a distance D from the point of projection, and at a time T from the instant of projection; we have, to determine D and T, the equations

$$D \cos \alpha = v'T \cos i, \quad D \sin \alpha = v'T \sin i - g\frac{T^2}{2};$$

$$\therefore \; D = \frac{2v'^2 \cos i \sin (i - \alpha)}{g \cos^2 \alpha},$$

$$\text{and } \; T = \frac{2v' \sin (i - \alpha)}{g \cos \alpha}.$$

The range on this plane, for a given velocity of projection, will be greatest when i is such that $\cos (2i - \alpha) = 0$; which gives $i = \frac{\alpha}{2} + \frac{\pi}{4}$.

III. *The motion of a particle acted on by a force which always tends to or from a fixed point.*

25. If a particle be projected in any direction, and be acted on by a force whose line of action always passes through a fixed point; the path of the particle will be wholly in the plane passing through the fixed point and the straight line along which the particle was initially projected. Let this plane be taken for the plane of reference.

The simplest way of expressing the motion by equations seems to be to use the radial and transversal accelerations of the particle's motion (art. 8, cor. 2). Let r, θ be the polar co-ordinates of the particle at the end of the time t measured

from a fixed instant, the point to or from which the force tends being the pole; and let P denote the accelerating effect of the force on the particle at the end of the same time, in direction towards the pole. We have

$$\left.\begin{aligned} \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 &= -P \\ \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) &= 0 \end{aligned}\right\} \dots\dots\dots\dots\dots (1)$$

the equations of motion of the particle.

The motion may be expressed by other equations in which the accelerating effect of the force is estimated differently. Thus, if x, y be the co-ordinates of the particle at the end of the time t, referred to rectangular axes through the center of force; then, considering the accelerations in the directions of the axes,

$$\frac{d^2x}{dt^2} = -P\frac{x}{r}, \quad \frac{d^2y}{dt^2} = -P\frac{y}{r} \dots\dots\dots\dots (2)$$

Or again, if the accelerating effects of the force in directions of the tangent and normal to the path be taken, we have

$$\frac{d^2s}{dt^2} = -P\frac{dr}{ds}, \quad \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 = P\frac{r\,d\theta}{ds} \dots\dots\dots\dots (3)$$

where s is the length of path intercepted between the particle and a fixed point in the path, and ρ is the radius of curvature of the path at the point where the particle is situated.

Each of these pairs of equations represents fully the motion.

The point to or from which the force always tends is called the center of force. The path of the particle is called a central orbit. A point in the orbit where the tangent is perpendicular to the radius-vector, is called an apse; the radius-vector at such a point, an apsidal distance; and the angle between two consecutive apsidal distances, the apsidal angle of the orbit.

26. *If a particle move in a central orbit, the sectorial areas swept out by its radius-vector in different times are proportional to the times.*

For from the second of equations (1)

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0,$$

whence by integration

$$r^2\frac{d\theta}{dt} = h,$$

in which h is an invariable quantity.

Now if A be the sectorial area swept out by the radius-vector in the time t, $dA = \frac{1}{2}r^2\,d\theta$,

$$\therefore \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}h,$$

$$\text{and } A = \frac{1}{2}ht,$$

if A be supposed to begin at the instant from which t is measured.

In the same way, if A' be the area swept out by the radius-vector in the time t', $A' = \frac{1}{2}ht'$;

$$\therefore A : A' = t : t'.$$

COR. 1. If ds be the elementary arc of the orbit corresponding to the elementary area dA, and p be the perpendicular drawn from the center of force on the tangent to the orbit at the extremity of ds; $dA = \frac{1}{2}p\,ds$;

$$\therefore 2\frac{dA}{dt} = p\frac{ds}{dt} = pv,$$

if v be the velocity of the particle. Hence

$$v = \frac{h}{p}.$$

This shews that *the velocity at any point of a central orbit is inversely proportional to the perpendicular drawn from the center of force on the tangent to the orbit at the point.*

If $\frac{1}{r} = u$, then $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$, and therefore,

$$v = h\sqrt{u^2 + \left(\frac{du}{d\theta}\right)^2}.$$

This expression might have been otherwise proved; for

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2.$$

But $\frac{d\theta}{dt} = hu^2$, and $\frac{dr}{dt} = \frac{d}{d\theta}\left(\frac{1}{u}\right)\frac{d\theta}{dt} = -h\frac{du}{d\theta}$;

$$\therefore\ v^2 = h^2\left(u^2 + \frac{du^2}{d\theta^2}\right).$$

Cor. 2. *If a particle move in such a manner that the sectorial areas swept out in different times by a straight line, drawn from it to a fixed point, are proportional to the times; the line of action of the force which acts on the particle passes through the fixed point.*

For if A be the area swept out in the time t, then

$$A = Kt,$$

in which K is invariable;

$$\therefore\ \frac{dA}{dt} \text{ or } r^2\frac{d\theta}{dt} = K;$$

$$\text{and } \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0:$$

that is, the force produces no acceleration of motion in a direction transverse to the radius-vector. Hence the force acts entirely along the radius-vector.

27. *To find the velocity of a particle in a central orbit in terms of the accelerating effect of the force acting on it.*

Since $v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$, and $\frac{d\theta}{dt} = \frac{h}{r^2}$;

$$\therefore\ v^2 = \left(\frac{dr}{dt}\right)^2 + \frac{h^2}{r^2}.$$

And differentiating this relative to t, we get

$$v\frac{dv}{dt} = \frac{dr}{dt}\left(\frac{d^2r}{dt^2} - \frac{h^2}{r^3}\right)$$

$$= \frac{dr}{dt}\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}$$

$$= -\frac{dr}{dt}P,$$

by the first of equations (1). Therefore

$$v\frac{dv}{dr} = -P,$$

$$\text{and } v^2 = C - 2\int Pdr,$$

C being an invariable quantity. Now suppose that v' is the velocity of the particle when it is at the distance r' from the center of force; then $v'^2 = C - 2\int_{r'} Pdr$;

$$\therefore\ v^2 = v'^2 - 2\int_{r'}^{r} Pdr.$$

In this P is supposed to depend only on r.

Hence the velocity of a particle at any point of a central orbit depends only on its distance from the center of force, and is therefore the same at all points of the orbit which are at the same distance from the center of force.

Hence also if any number of particles move about the same center of force, and if at certain points of their orbits equidistant from the center of force their velocities be equal to one another; their velocities at all points of their orbits equidistant from the center of force will be equal to one another.

The last equation may be readily deduced from the first of equations (3), viz.:

$$v\frac{dv}{ds}\left(=\frac{d^2s}{dt^2}\right) = -P\frac{dr}{ds};$$

$$\therefore\ v\frac{dv}{dr} = -P.$$

COR. 1. By differentiating relative to r the equation $v^2 = \frac{h^2}{p^2}$, we get

$$v\frac{dv}{dr} = -\frac{h^2}{p^3}\frac{dp}{dr} = -\frac{v^2}{p}\frac{dp}{dr};$$

$$\therefore\ P = \frac{v^2}{p}\frac{dp}{dr};$$

$$\text{or } v^2 = Pp\frac{dr}{dp}.$$

But $p\frac{dr}{dp}$ is = half the chord of curvature drawn through the center of force. Therefore

$$v^2 = 2P \times \text{one fourth of the chord of curvature.}$$

Hence *the velocity of a particle at any point of a central orbit is the same as it would acquire by falling freely from rest down one fourth of the chord of curvature at the point drawn through the center of force, under the action of a constant force whose intensity is equal to the intensity of the central force at the point.* (see art. 16.)

This expression for v^2 is immediately found from the second of equations (3), which gives

$$\left(\frac{ds}{dt}\right)^2 = 2P.\frac{\rho}{2}\frac{r\,d\theta}{ds};$$

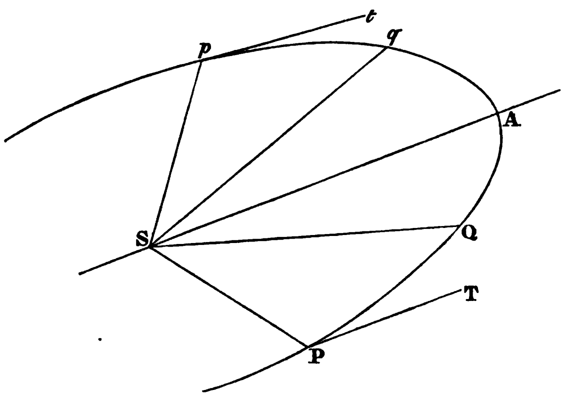
and $\frac{\rho}{2}\frac{r\,d\theta}{ds}$ = one fourth of the chord of curvature through the pole.

COR. 2. Since $v^2 = h^2\left(u^2 + \frac{du^2}{d\theta^2}\right)$, and since the velocity is the same at all points of the orbit which are equidistant from the center of force; therefore $\left(\frac{du}{d\theta}\right)^2$ is also the same at all such points. Hence at all points which are equidistant

from the center of force the inclination of the tangent to the radius-vector is the same. It follows from this, that,

If in a central orbit there be drawn from the center of force two radii-vectors equal to one another and meeting the intercepted portion of the orbit in angles which are either both acute, or both obtuse, the straight line bisecting the angle between the radii-vectors will divide the orbit into two perfectly equal and similar parts.

Let SP, Sp be equal radii-vectors in the orbit PAp, and meeting the intercepted portion in the angles SPT, Spt which are either both acute, or both obtuse. Bisect PSp by the straight line SA, and let SQ, Sq be any radii-vectors making equal angles with SA and on opposite sides of it. There-



fore the angles PSQ, pSq are equal. Now $\left(\frac{du}{d\theta}\right)^2$, being a function of u only, may be denoted by $f(u)$; and if at P, $\frac{du}{d\theta}$ be denoted by $+\sqrt{f(u)}$, then at p it will $= -\sqrt{f(u)}$.

Hence by integration we have

$$\text{angle } PSQ = \int_{\frac{1}{SP}}^{\frac{1}{SQ}} \frac{du}{\sqrt{f(u)}}, \quad \text{angle } pSq = -\int_{\frac{1}{Sq}}^{\frac{1}{Sp}} \frac{du}{\sqrt{f(u)}}.$$

And since the angles PSQ, pSq are equal to one another;

$$\therefore \int_{\frac{1}{SP}}^{\frac{1}{SQ}} \frac{du}{\sqrt{f(u)}} + \int_{\frac{1}{Sq}}^{\frac{1}{Sp}} \frac{du}{-\sqrt{f(u)}} = 0.$$

But $SP = Sp$; therefore $SQ = Sq$. Hence straight lines drawn from S to the orbit on opposite sides of SA and making equal angles with it are equal to one another; consequently the straight line SA divides the orbit into two parts which are equal and similar.

It is evident that the point A in which SA meets the orbit is an apse: for since the orbit is necessarily continuous and not broken at any point, therefore SA meets the curve at right angles.

If the force in a central orbit tend either continually to, or continually from the center of force, every apsidal line divides the orbit into parts which are equal and similar.

For such an orbit is either concave or convex towards the center of force at every point; consequently a straight line may be drawn near the apse perpendicular to the apsidal distance, to cut the orbit in two points on opposite sides of the apse. Now as the perpendicular straight line is made continually to approach and ultimately to coincide with the tangent to the orbit at the apse, the two points will ultimately be equidistant from the apse, and therefore also from the center of force; and the apsidal distance will ultimately bisect the angle formed by straight lines drawn from the center of force to the points. Hence by what goes before the apsidal line divides the orbit symmetrically.

This may also be directly proved thus:—Since the orbit is either concave at every point or convex, therefore, not only is $\frac{du}{d\theta} = 0$ at an apse, but it changes its algebraical sign in passing through the apse. Hence if on one side of the apse it be represented by $+\sqrt{f(u)}$, on the other side it must be represented by $-\sqrt{f(u)}$. Let then u' and $u_{,}$ be inverse radii-

vectors on opposite sides of the apsidal distance and making equal angles with it, and let a be the inverse apsidal distance. We have

$$\text{angle between } \frac{1}{u'} \text{ and } \frac{1}{a} = \int_a^{u'} \frac{du}{\sqrt{f(u)}},$$

$$\text{angle between } \frac{1}{u_{\prime}} \text{ and } \frac{1}{a} = -\int_{u_{\prime}}^{a} \frac{du}{\sqrt{f(u)}};$$

$$\therefore \quad \int_a^{u'} \frac{du}{\sqrt{f(u)}} + \int_{u_{\prime}}^{a} \frac{du}{\sqrt{f(u)}} = 0;$$

$$\text{or} \quad \int_{u_{\prime}}^{u'} \frac{du}{\sqrt{f(u)}} = 0, \text{ identically};$$

$$\therefore \quad u_{\prime} = u';$$

that is, lines which are inclined to the apsidal line at equal angles on opposite sides of it are equal to one another; and therefore, the apsidal line cuts the orbit symmetrically.

COR. 3. *In a central orbit, if the force tend either continually to or continually from the center of force, there cannot be more than two different apsidal distances, nor more than one apsidal angle.*

Let A, a be two adjoining apsidal distances in an orbit, and let α be the apsidal angle between them. Since A divides the orbit symmetrically, therefore going round the orbit in direction from a to A, the next apsidal distance which occurs after passing A must be one $(a') = a$ making with A an apsidal angle $= \alpha$. Again, going round in the same direction, after passing a' the next apsidal distance which occurs is one $= A$ and making with a' an apsidal angle $= \alpha$; because a' divides the orbit symmetrically. Hence it is clear that the apsidal distances occur alternately equal to A and a, and every adjoining two of them make with one another an angle $= \alpha$.

28. *To find the differential equation of a central orbit.*

We have $\dfrac{dr}{dt} = \dfrac{d}{d\theta}\left(\dfrac{1}{u}\right)\dfrac{d\theta}{dt} = -h\dfrac{du}{d\theta}$, since $\dfrac{d\theta}{dt} = hu^2$,

$$\text{and } \frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}.$$

Substituting in the equation $\dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 = -P$,

$$\text{we get } -h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4 = -P;$$

$$\therefore \frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$$

the differential equation sought.

This equation also gives the nature of the central force, when the orbit is known. It may be put in another form which is sometimes convenient; from the equation $v^2 = \dfrac{h^2}{p^2}$,

$$v\frac{dv}{dr} = -\frac{h^2}{p^3}\frac{dp}{dr}.$$

$$\text{But } v\frac{dv}{dr} = -P;$$

$$\therefore P = \frac{h^2dp}{p^3dr}.$$

29. *A particle describes a circle under the action of a force tending to the center; to find the intensity of the force.*

From the formulæ

$$v^2 = h^2\left(u^2 + \frac{du^2}{d\theta^2}\right) \text{ and } P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right),$$

we have, since u is constant,

$$v^2 = h^2u^2, \quad P = h^2u^3.$$

Hence the accelerating effect of the force $= v^2u$ or $\dfrac{v^2}{r}$.

And the time of describing any angle θ is $= \dfrac{\theta r}{v}$.

30. *A particle describes an ellipse under the action of a force tending to the center; to find the intensity of the force and the velocity of the particle at any point of the orbit, also the*

time occupied by the passage of the particle from one point to another.

Let the equation of the ellipse be $u^2 = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}$, in which a and b are the semi-axes, and θ is measured from the apse at the extremity of a. This equation may be written

$$u^2 = \tfrac{1}{2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) + \tfrac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\cos 2\theta;$$

$$\therefore\ u\frac{du}{d\theta} = -\tfrac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\sin 2\theta,$$

squaring each side of these equations and adding, we get

$$u^2\left(u^2 + \frac{du^2}{d\theta^2}\right) = \tfrac{1}{4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2 + \tfrac{1}{4}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)^2 + \tfrac{1}{2}\left(\frac{1}{a^4} - \frac{1}{b^4}\right)\cos 2\theta$$

$$= \left(\frac{1}{a^2} + \frac{1}{b^2}\right)u^2 - \frac{1}{a^2b^2},$$

$$\text{and } u^2 + \left(\frac{du}{d\theta}\right)^2 = \left(\frac{1}{a^2} + \frac{1}{b^2}\right) - \frac{u^{-2}}{a^2b^2};$$

$$\therefore\ u + \frac{d^2u}{d\theta^2} = \frac{u^{-3}}{a^2b^2}.$$

Hence $P = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = \frac{h^2u^{-1}}{a^2b^2} = \mu r,$

in which μ is $= \frac{h^2}{a^2b^2}$, and is the same for all points in the orbit; it is the accelerating effect of the force at unit of distance from the center to which the force tends.

The intensity of the force at any point of the orbit is therefore proportional to the distance of the particle from the center of force.

The velocity of the particle $= h\sqrt{u^2 + \left(\frac{du}{d\theta}\right)^2}$

$$= ab\sqrt{\mu}\sqrt{\frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}}$$

$$= \sqrt{\mu\,(a^2 + b^2 - r^2)}$$

or $= \sqrt{\mu} \times$ (semi-diameter which is conjugate to r).

The time of the particle describing any part of the orbit is to be found from the formula

$$r^2 \frac{d\theta}{dt} = h, \text{ in the present case } = ab\sqrt{\mu},$$

$$\therefore dt = \frac{r^2 d\theta}{ab\sqrt{\mu}} = \frac{1}{ab\sqrt{\mu}} \cdot \frac{d\theta}{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}},$$

$$\text{and } t = \frac{1}{ab\sqrt{\mu}} \int \frac{\sec^2\theta \,.\, d\theta}{\dfrac{1}{a^2} + \dfrac{1}{b^2}\tan^2\theta}$$

$$= \frac{1}{\sqrt{\mu}} \tan^{-1}\left(\frac{a\tan\theta}{b}\right),$$

no correction being required if θ and t be supposed to begin together. This gives the time occupied by the particle in describing an angle θ measured from the apse at the extremity of a.

Also the time occupied by the passage of the particle from a point of the orbit whose angle-vector is θ, to another point whose angle-vector is θ', is

$$= \frac{1}{\sqrt{\mu}} \left\{ \tan^{-1}\left(\frac{a\tan\theta'}{b}\right) - \tan^{-1}\left(\frac{a\tan\theta}{b}\right) \right\}.$$

Hence the time of a complete revolution of the particle from any point to the same point again is

$$= \frac{2\pi}{\sqrt{\mu}}.$$

This is independent of the dimensions of the orbit; therefore the periodic times in ellipses, about the same center of force in their common center, are equal to one another. The periodic time may be found without referring to the general expression for the time of describing any angle.

The periodic time

$$= \frac{\text{area of the orbit}}{\text{area described by the radius-vector in a unit of time}}$$

$$= \frac{2 \times \text{area of the orbit}}{h}$$

$$= \frac{2\pi ab}{\sqrt{\mu}\,.\,ab} = \frac{2\pi}{\sqrt{\mu}}.$$

31. *A particle describes a conic section about a center of force in one of the foci; to find the intensity of the force at any point of the orbit, the velocity of the particle, and the time of describing any angle measured from the nearer apse.*

The equation of a conic section referred to the focus as pole is

$$u = \frac{1}{k}(1 + e\cos\theta),$$

in which k is the semi-latus rectum, and e a quantity less than, equal to, or greater than unity, according as the curve is an ellipse, parabola, or hyperbola;

$$\therefore \frac{du}{d\theta} = -\frac{e}{k}\sin\theta.$$

$$\text{Hence } \left(u - \frac{1}{k}\right)^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{e^2}{k^2},$$

$$\text{or } u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{2u}{k} - \frac{1-e^2}{k^2}.$$

Whence by differentiating we have

$$u + \frac{d^2u}{d\theta^2} = \frac{1}{k}.$$

$$\text{Therefore } P = \frac{h^2u^2}{k} = \frac{\mu}{r^2},$$

where $\mu = \frac{h^2}{k}$ = the accelerating effect of the force at distance unity from the center of force.

Hence at any point of the orbit the intensity of the force on the particle is inversely proportional to the square of the particle's distance from the center of force.

To find the velocity of the particle at any point of its orbit, we have

$$v^2 = h^2\left(u^2 + \frac{du^2}{d\theta^2}\right) = \mu\left(\frac{2}{r} - \frac{1-e^2}{k}\right),$$

$\therefore$ in an ellipse, $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$, a being the semi-axis major,

in a parabola, $v^2 = \frac{2\mu}{r}$,

and in a hyperbola, $v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$, a being the semi-axis transverse.

If a particle described a circle of radius r under the action of a force tending to its center and whose accelerating effect $= \frac{\mu}{r^2}$, its velocity would be $= \sqrt{\frac{\mu}{r^2}\cdot r}$ (art. 29) $= \sqrt{\frac{\mu}{r}}$. Hence the square of the velocity of a particle moving in a parabola is at any point of the path equal to twice the square of the velocity in a circle at the same distance, the square of the velocity of a particle moving in an ellipse is less, and that of a particle moving in a hyperbola is greater.

The time of the particle's describing any angle, measured from the nearer apse is to be obtained from the formula

$$\frac{r^2 d\theta}{dt} = h = \sqrt{\mu k};$$

$$\therefore\ dt = \frac{r^2 d\theta}{\sqrt{\mu k}} = \sqrt{\frac{k^3}{\mu}}\ \frac{d\theta}{(1 + e\cos\theta)^2}.$$

In order to integrate this, put $Q = \frac{\sin\theta}{1 + e\cos\theta}$,

$$\text{then } \frac{dQ}{d\theta} = \frac{\cos\theta + e}{(1 + e\cos\theta)^2} = \frac{\frac{1}{e}(1 + e\cos\theta) - \frac{1-e^2}{e}}{(1 + e\cos\theta)^2}$$

$$= \frac{1}{e}\,\frac{1}{1 + e\cos\theta} - \frac{1-e^2}{e}\,\frac{1}{(1 + e\cos\theta)^2};$$

$$\therefore \int \frac{d\theta}{(1+e\cos\theta)^2} = -\frac{e}{1-e^2}Q + \frac{1}{1-e^2}\int\frac{d\theta}{1+e\cos\theta}$$

$$= -\frac{e}{1-e^2}\frac{\sin\theta}{1+e\cos\theta} + \frac{1}{1-e^2}\int\frac{\sec^2\frac{\theta}{2}\,d\theta}{1+e+(1-e)\tan^2\frac{\theta}{2}}$$

$$= -\frac{e}{1-e^2}\frac{\sin\theta}{1+e\cos\theta} + \frac{2}{(1-e^2)^{\frac{3}{2}}}\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right),$$

if e be less than unity;

$$= \frac{e}{e^2-1}\frac{\sin\theta}{1+e\cos\theta} - \frac{1}{(e^2-1)^{\frac{3}{2}}}\log\left\{\frac{\sqrt{e+1}\cos\frac{\theta}{2}+\sqrt{e-1}\sin\frac{\theta}{2}}{\sqrt{e+1}\cos\frac{\theta}{2}-\sqrt{e-1}\sin\frac{\theta}{2}}\right\},$$

if e be greater than unity; and if e be equal to unity the integral becomes

$$\int\frac{d\theta}{(1+\cos\theta)^2} = \tfrac{1}{4}\int\sec^4\frac{\theta}{2}\,d\theta = \tfrac{1}{2}\int\left(1+\tan^2\frac{\theta}{2}\right)d\left(\tan\frac{\theta}{2}\right)$$

$$= \tfrac{1}{2}\left(\tan\frac{\theta}{2} + \tfrac{1}{3}\tan^3\frac{\theta}{2}\right).$$

Hence the time of describing an angle θ about the pole measured from the nearer apse is

$$\text{in the ellipse} = \sqrt{\frac{a^3}{\mu}}\left\{2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - e\sqrt{1-e^2}\frac{\sin\theta}{1+e\cos\theta}\right\},$$

$$\text{in the parabola} = \tfrac{1}{2}\sqrt{\frac{k^3}{\mu}}\left(\tan\frac{\theta}{2} + \tfrac{1}{3}\tan^3\frac{\theta}{2}\right),$$

and in the hyperbola

$$= \sqrt{\frac{a^3}{\mu}}\left\{\log\left(\frac{\sqrt{e+1}\cos\theta-\sqrt{e-1}\sin\theta}{\sqrt{e+1}\cos\theta+\sqrt{e-1}\sin\theta}\right) + e\sqrt{e^2-1}\frac{\sin\theta}{1+e\cos\theta}\right\}.$$

COR. The periodic time in an ellipse is therefore

$$= 2\pi\sqrt{\frac{a^3}{\mu}}.$$

This shews that *in different elliptic orbits the squares of the periodic times are proportional directly to the cubes of the mean distances, and inversely to the accelerative intensities of the forces at the unit of distance from the centers of force.*

The periodic time may be found directly; for it

$$=\frac{2\times\text{area of the ellipse}}{h}=\frac{2\pi ab}{\sqrt{\mu k}}$$

$$=2\pi\sqrt{\frac{a^3}{\mu}}.$$

32. *To find the law of force tending to the pole under which a particle may describe an equiangular spiral.*

Let the equation of the spiral be $r=a\,\epsilon^{\theta\cot\alpha}$, α being the inclination of the tangent to the radius-vector at every point.

We have

$$u=\frac{1}{a}\,\epsilon^{-\theta\cot\alpha};$$

$$\therefore\ \frac{du}{d\theta}=-\frac{\cot\alpha}{a}\,\epsilon^{-\theta\cot\alpha}=-u\cot\alpha,$$

and $\left(\dfrac{du}{d\theta}\right)^2+u^2=u^2\operatorname{cosec}^2\alpha.$

$$\text{Hence } \frac{d^2u}{d\theta^2}+u=u\operatorname{cosec}^2\alpha,$$

$$\text{and } P=h^2u^3\operatorname{cosec}^2\alpha=\frac{\mu}{r^3};$$

μ being the accelerating effect of the force at the unit of distance from the center of force. The force therefore varies inversely as the cube of the distance.

The velocity at a point of the orbit

$$=hu\operatorname{cosec}\alpha=\frac{\sqrt{\mu}}{r},$$

and therefore it varies inversely as the distance.

Also the time of describing an arc of the spiral

$$=\frac{1}{h}\int r^2\,d\theta=\frac{a^2\operatorname{cosec}\alpha}{\sqrt{\mu}}\int\epsilon^{2\theta\cot\alpha}\,d\theta=\frac{a^2\sec\alpha}{2\sqrt{\mu}}\cdot\epsilon^{2\theta\cot\alpha}+\text{a constant}.$$

Hence the time of describing an arc, the radii-vectors of whose extremities are r and r', is

$$=\frac{\sec a}{2\sqrt{\mu}}(r^2 - r'^2).$$

33. *A particle, being projected from a given point in a given direction with a given velocity, is acted on by a force tending to a fixed center, whose intensity at any point is proportional to the distance of the point from the center of force; to find the orbit described by the particle.*

Let the center of force be taken for pole, and the straight line drawn to the point of projection for prime radius. Let r' be the distance of the point of projection from the center of force, v' the velocity of projection, and i the angle which the direction of projection makes with r'. The accelerating effect of the force on the particle when it is at distance r from the center of force may be represented by μr. Putting this for P in the equation of article 28, we have

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2u^3},$$

in which clearly

$$h = v'r' \sin i.$$

Multiplying each side by $2\dfrac{du}{d\theta}$ and integrating

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = C - \frac{\mu}{h^2u^2},$$

C being a constant. This is equivalent to

$$v^2 = Ch^2 - \frac{\mu}{u^2}.$$

Now when $u = \dfrac{1}{r'}$, $v = v'$; $\quad\therefore\quad v'^2 = Ch^2 - \mu r'^2$;

and $\therefore$ $$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{v'^2 + \mu r'^2}{h^2} - \frac{\mu}{h^2u^2}.$$

Hence $\left(\dfrac{du}{d\theta}\right)^2 = \dfrac{1}{u^2}\left(\dfrac{v'^2+\mu r'^2}{h^2}u^2 - \dfrac{\mu}{h^2} - u^4\right)$

$$= \frac{1}{u^2}\left\{\left(\frac{v'^2+\mu r'^2}{2h^2}\right)^2 - \frac{\mu}{h^2} - \left(u^2 - \frac{v'^2+\mu r'^2}{2h^2}\right)^2\right\};$$

$$\therefore \frac{u\,du}{\sqrt{\left(\frac{v'^2+\mu r'^2}{2h^2}\right)^2 - \frac{\mu}{h^2} - \left(u^2 - \frac{v'^2+\mu r'^2}{2h^2}\right)^2}} = d\theta;$$

in which either sign of the radical may be taken. Taking the negative sign, integrating, and using the constant ϖ,

$$\cos^{-1}\left\{\frac{u^2 - \frac{v'^2+\mu r'^2}{2h^2}}{\sqrt{\left(\frac{v'^2+\mu r'^2}{2h^2}\right)^2 - \frac{\mu}{h^2}}}\right\} = 2(\theta - \varpi).$$

To find ϖ, we have, since $u = \dfrac{1}{r'}$ when $\theta = 0$,

$$\cos 2\varpi = \frac{\frac{1}{r'} - \frac{v'^2+\mu r'^2}{2h^2}}{\sqrt{\left(\frac{v'^2+\mu r'^2}{2h^2}\right)^2 - \frac{\mu}{h^2}}}$$

$$= \frac{2v'^2\sin^2 i - v'^2 - \mu r'^2}{\sqrt{(v'^2+\mu r'^2)^2 - 4\mu v'^2 r'^2 \sin^2 i}}.$$

Hence the equation of the orbit is

$$u^2 = \frac{v'^2+\mu r'^2}{2h^2} + \sqrt{\left(\frac{v'^2+\mu r'^2}{2h^2}\right)^2 - \frac{\mu}{h^2}}\cos 2(\theta - \varpi);$$

or, putting for h its value,

$$\frac{1}{r^2} = \frac{v'^2+\mu r'^2}{2v'^2 r'^2 \sin^2 i}\left\{1 + \sqrt{1 - \frac{4\mu v'^2 r'^2 \sin^2 i}{(v'^2+\mu r'^2)^2}}\,.\cos 2(\theta - \varpi)\right\},$$

which is the equation of an ellipse whose center coincides with the center of force, and the angle-vector of whose apse is ϖ, or

$$-\frac{1}{2}\cos^{-1}\left\{\frac{2v'^2\sin^2 i - v'^2 - \mu r'^2}{\sqrt{(v'^2+\mu r'^2)^2 - 4\mu v'^2 r'^2 \sin^2 i}}\right\}$$

$$= \frac{1}{2}\cot^{-1}\left(\cot 2i + \frac{\mu r'^2}{v'^2}\operatorname{cosec} 2i\right).$$

34. *A particle, projected from a given point in a given direction with a given velocity, is acted on by a force which tends to a fixed center and whose accelerating effect on the particle at a point is inversely proportional to the square of the distance of the point from the center of force; to find the orbit described by the particle.*

Let the center of force be taken for pole, and the straight line drawn from the center of force to the point of projection for prime radius. Also let r', v' and i, denote respectively the distance, velocity, and angle of projection; and let μ denote the accelerating effect of the force at distance unity from the center of force. We have

$$h = v'r'\sin i, \qquad P = \frac{\mu}{r^2} = \mu u^2;$$

therefore the differential equation of the orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2};$$

or, as it may be written,

$$\frac{d^2}{d\theta^2}\left(u - \frac{\mu}{h^2}\right) + \left(u - \frac{\mu}{h^2}\right) = 0.$$

Multiplying by $2\frac{d}{d\theta}\left(u - \frac{\mu}{h^2}\right)$, and integrating, we get

$$\left\{\frac{d\left(u - \frac{\mu}{h^2}\right)}{d\theta}\right\}^2 + \left(u - \frac{\mu}{h^2}\right)^2 = C, \text{ a constant.}$$

Now when $u = \frac{1}{r'}$, the velocity is $= v'$, and hence

$$v'^2 = \frac{2\mu}{r'} - \frac{\mu^2}{h^2} + Ch^2;$$

$$\therefore\ C = \frac{\mu^2}{h^4} - \frac{\dfrac{2\mu}{r'} - v'^2}{h^2}.$$

For the sake of shortness let this be written $\dfrac{\mu^2 e^2}{h^4}$, where

$$e = \sqrt{1 - \frac{h^2}{\mu^2}\left(\frac{2\mu}{r'} - v'^2\right)} = \sqrt{1 - \frac{v'^2 r'^2 \sin^2 i}{\mu^2}\left(\frac{2\mu}{r'} - v'^2\right)}.$$

$$\text{Hence } \frac{d}{d\theta}\left(u - \frac{\mu}{h^2}\right) = \sqrt{\frac{\mu^2 e^2}{h^4} - \left(u - \frac{\mu}{h^2}\right)^2},$$

$$\text{and } \frac{d\left(u - \dfrac{\mu}{h^2}\right)}{\sqrt{\dfrac{\mu^2 e^2}{h^4} - \left(u - \dfrac{\mu}{h^2}\right)^2}} = d\theta,$$

in which either sign of the radical may be taken. Integrating, we get

$$\cos^{-1}\left(\frac{u - \dfrac{\mu}{h^2}}{\dfrac{\mu e}{h^2}}\right) = \theta - \varpi,$$

ϖ being a constant, to determine which we have, since $u = \dfrac{1}{r'}$, when $\theta = 0$,

$$\frac{1}{r'} - \frac{\mu}{h^2} = \frac{\mu e}{h^2}\cos\varpi;$$

$$\therefore\ \varpi = \cos^{-1}\left\{\frac{v'^2 r' \sin^2 i - \mu}{\sqrt{\mu^2 - v'^2 r'^2 \sin^2 i\left(\dfrac{2\mu}{r'} - v'^2\right)}}\right\}$$

$$= \cot^{-1}\left(\tan i - \frac{2\mu}{v'^2 r'}\operatorname{cosec} 2i\right).$$

Hence the equation of the orbit is

$$\frac{1}{r} = \frac{\mu}{v'^2 r'^2 \sin^2 i}\left\{1 + e\cos(\theta - \varpi)\right\},$$

in which e and ϖ are determined above.

This is the equation of a conic section referred to a focus as pole; the curve is an ellipse, parabola, or hyperbola, according as e is less than, equal to, or greater than unity; that is, according as $\frac{2\mu}{r'} - v'^2$ is positive, zero, or negative. The latus rectum is equal to $\frac{2}{\mu} v'^2 r'^2 \sin^2 i$.

In the case of an ellipse, if a, b be the semi-axes, we have

$$a(1-e^2) = \frac{v'^2 r'^2 \sin^2 i}{\mu}, \quad 1 - e^2 = \frac{v'^2 r'^2 \sin^2 i}{\mu^2} \left(\frac{2\mu}{r'} - v'^2\right),$$

$$\therefore a = \frac{\mu}{\frac{2\mu}{r'} - v'^2}, \quad b = \frac{v' r' \sin i}{\sqrt{\frac{2\mu}{r'} - v'^2}}.$$

Since the angle of projection does not enter into the expression for a, therefore the magnitude of the semi-axis major is independent of the direction in which the particle was projected.

35. *A particle describes an orbit nearly circular about a center of attractive force; having given the law of force, to find approximately the equation of the orbit.*

Let $f(r)$ denote the accelerating effect of the force at distance r from the center of force; let a be an apsidal distance in the orbit, and let the velocity of the particle when at this apse be to the velocity in a circle at the same distance as $1 + n : 1$, in which n is a very small fraction; also let $a + x$ be the radius-vector corresponding to the angle-vector θ, x being very small compared with a.

Omitting higher powers than the first of the small quantities n and $\frac{x}{a}$, we have

$$u = \frac{1}{a+x} = \frac{1}{a} - \frac{x}{a^2},$$

$$\frac{f(r)}{u^2} = f(a+x)(a+x)^2 = a^2 f(a) + x\left\{2a f(a) + a^2 f'(a)\right\},$$

in which f' denotes the derivative of f,

and $h^2 = a^2 \times (\text{velocity})^2$ at the apse

$= a^2(1+n)^2 af(a) = a^3 f(a)(1+2n)$.

$$\text{Hence } \frac{f(r)}{h^2u^2} = \frac{a^2 f(a) + ax f(a)\left\{2 + \dfrac{af'(a)}{f(a)}\right\}}{a^3 f(a)(1+2n)}$$

$$= \frac{1}{a} - \frac{2n}{a} + \frac{x}{a^2}\left\{2 + \frac{af'(a)}{f(a)}\right\}.$$

Substituting in the equation $\dfrac{d^2u}{d\theta^2} + u - \dfrac{P}{h^2u^2} = 0$, we have

$$-\frac{1}{a^2}\frac{d^2x}{d\theta^2} + \frac{1}{a} - \frac{x}{a^2} = \frac{1}{a} - \frac{2n}{a} + \frac{x}{a^2}\left\{2 + \frac{af'(a)}{f(a)}\right\}.$$

$$\text{Whence } \frac{d^2x}{d\theta^2} + x\left\{3 + \frac{af'(a)}{f(a)}\right\} = 2na.$$

The solution of this equation is

$$x = \frac{2na}{3 + \dfrac{af'(a)}{f(a)}} + A\cos\sqrt{3 + \frac{af'(a)}{f(a)}}\,(\theta - \varpi),$$

ϖ being the angle-vector of the apse, and A a quantity to be found. Now when $\theta = \varpi$, $x = 0$;

$$\therefore A = -\frac{2na}{3 + \dfrac{af'(a)}{f(a)}}.$$

Hence the equation of the orbit is

$$r = a\left[1 + \frac{2n}{3 + \dfrac{af'(a)}{f(a)}}\left\{1 - \cos\left(\sqrt{3 + \frac{af'(a)}{f(a)}}\,(\theta - \varpi)\right)\right\}\right].$$

The angles-vector of the apsides in the orbit are manifestly the values of θ which satisfy the equation

$$\sin\{k(\theta - \varpi)\} = 0,$$

in which k is put for $\sqrt{3+\dfrac{af'(a)}{f(a)}}$; these angles are

$$\varpi,\ \varpi+\frac{\pi}{k},\ \ \varpi+\frac{2\pi}{k},\ \ \varpi+\frac{3\pi}{k},\ \text{\&c.}$$

Hence the apsidal angle is $=\dfrac{\pi}{\sqrt{3+\dfrac{af'(a)}{f(a)}}}$.

The apsidal distances are a, and $a\left\{1+\dfrac{4n}{3+\dfrac{af'(a)}{f(a)}}\right\}$.

COR. 1. If the force vary directly as the distance, we may put $f(a)=\mu a$, and $\therefore\ \dfrac{af'(a)}{f(a)}=1$; hence the apsidal angle $=\dfrac{\pi}{\sqrt{3+1}}=\dfrac{\pi}{2}$.

If the force vary inversely as the square of the distance, $f(a)=\dfrac{\mu}{a^2}$, and the apsidal angle $=\dfrac{\pi}{\sqrt{3-2}}=\pi$.

Generally, if the force vary as the i^{th} power of the distance, the apsidal angle $=\dfrac{\pi}{\sqrt{3+i}}$. In order that this may be possible i must not be less than -3; hence the force must not vary according to a higher inverse power of the distance than the third.

COR. 2. To find the law of force necessary in order that the apsidal angle in the orbit may be equal to a given angle, α suppose; we have

$$\frac{\pi}{\sqrt{3+\dfrac{af'(a)}{f(a)}}}=\alpha;$$

$$\therefore\ \frac{f'(a)}{f(a)}=\left(\frac{\pi^2}{\alpha^2}-3\right)\frac{1}{a}.$$

Integrating and using the constant μ,

$$f(a) = \mu\, a^{\frac{\pi^2}{a^2} - 3}$$

36. The foregoing principles lead to important conclusions with regard to the forces which act on the planets. Kepler first stated the following laws of the planetary motions, as results drawn from his observations :

(1) *The orbit described about the sun by each of the planets is an ellipse having the sun in one focus.*

(2) *The areas passed over in different lengths of time by a straight line drawn from the sun to any (the same) planet, are proportional to these lengths of time.*

(3) *The squares of the periodic times of different planets are proportional to the cubes of their mean distances from the sun, that is, of the semi-axes major of their elliptic orbits.*

If the sun and planets be considered material particles (which is not an extravagant supposition, when the smallness of their bodies compared to the immense distances between them is considered), it follows from the second of these laws (art. 26, cor. 2) that the force which acts on any planet is directed towards the sun; from the first (art. 31) that the accelerating effects of the forces acting on the same planet at different points of its orbit are inversely proportional to the squares of its distances from the sun; and from the third (art. 31, cor.) that the accelerating effects of the sun's attractive force on different planets are inversely proportional to the squares of their distances from the sun. Hence the accelerating effect of the sun's attraction on a planet may be represented by μr^{-2}, r being the distance of the planet from the sun and μ a quantity which is the same for all the planets.

The motions of the planets are found to accord with Kepler's laws during intervals of time which are not very long: but in long intervals discrepancies appear. Newton, who was the first to draw from Kepler's laws the full conclusions as to the nature of the forces acting on the planets, was likewise the first to shew that the discrepancies could be explained by supposing that each planet attracts every other

planet and the sun with a force whose accelerating effect at different distances follows the same law as that of the sun's attraction,—viz. the law of the inverse square of the distance,—but at the same distance is very much less than the accelerating effect of the sun's attractive force. He further explained the motions of the other bodies of the solar system by supposing that the same law of attraction applied to them; and he was at length led to infer the most general physical law yet known,—the Law of Universal Gravitation.—That *every particle of matter attracts every other particle with a force whose accelerating effect is proportional directly to the mass of the attracting particle and inversely to the square of the distance between the attracted and attracting particles.* This law has been abundantly confirmed since the days of Newton.

IV. *The motion of a particle acted on by a force whose direction is always parallel to a fixed plane.*

37. When a particle is acted on by a force parallel to a fixed plane, let the plane of xy be taken parallel to the fixed plane, and the axis of z perpendicular to it. The component accelerating effects of the force on the particle may be represented by X, Y parallel respectively to the axes of x, y, and by zero parallel to the axis of z. If x, y, z, denote the co-ordinates of the particle at the end of the time t measured from a fixed epoch, the equations of motion are

$$\frac{d^2x}{dt^2} = X, \qquad \frac{d^2y}{dt^2} = Y, \qquad \frac{d^2z}{dt^2} = 0.$$

From the last of these equations we have $\frac{dz}{dt} = \gamma$, a quantity independent of the particle's position and of the time. Again integrating,

$$z = \gamma t + c,$$

c being another quantity independent of x, y, z and t. Hence throughout the motion the component velocity parallel to the axis of z is constant, and the distance of the particle from the plane of xy at any instant is at once found. If

$\gamma = 0$, that is, if the particle be initially projected along or parallel to the plane of xy, the motion will be in or parallel to that plane. This case of motion therefore includes in it the most general kind of motion in a plane.

If γ be not zero, we have

$$\frac{dx}{dt} = \frac{dx}{dz}\,\frac{dz}{dt} = \gamma\frac{dx}{dz}, \text{ and } \frac{d^2x}{dt^2} = \gamma^2\,\frac{d^2x}{dz^2}.$$

$$\text{Similarly } \frac{d^2y}{dt^2} = \gamma^2\frac{d^2y}{dz^2}.$$

Substituting in the two first equations of motion

$$\gamma^2\frac{d^2x}{dz^2} = X, \quad \gamma^2\frac{d^2y}{dz^2} = Y.$$

To find the path when the forces are given, we must endeavour to obtain from these equations two integral equations, which will be the equations of two surfaces whose line of intersection is the path. If the path be given, these equations will determine the accelerating effect of the force acting parallel to a fixed plane requisite for the description.

If γ be zero, or if the two first equations of motion do not involve z, we have, multiplying the equations by $2\dfrac{dx}{dt}$, $2\dfrac{dy}{dt}$ respectively, and integrating,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\int(Xdx + Ydy);$$

the constant being supposed included in the sign of integration. If $Xdx + Ydy$ be a perfect differential this gives the velocity, or $\dfrac{ds}{dt}$ if ds be an element of the path.

Again, since

$$\frac{d^2x}{dt^2} = \frac{d^2x}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dx}{ds}\,\frac{d^2s}{dt^2}, \quad \frac{d^2y}{dt^2} = \frac{d^2y}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dy}{ds}\,\frac{d^2s}{dt^2};$$

therefore multiplying the equations of motion by $\frac{d^2x}{ds^2}$ and $\frac{d^2y}{ds^2}$ and adding, we get

$$X\frac{d^2x}{ds^2}+Y\frac{d^2y}{ds^2}=\left\{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2\right\}\left(\frac{ds}{dt}\right)^2;$$

$$\text{or } X\frac{d^2x}{ds^2}+Y\frac{d^2y}{ds^2}=2\left\{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2\right\}\int(Xdx+Ydy);$$

which is the differential equation either of the path itself (if $\gamma=0$) or of the projection of the path on the plane of xy. This equation might have been immediately obtained by using the normal resolution of acceleration.

When the path or its projection on the plane of xy is found, the equation

$$t=\int\frac{ds}{\sqrt{2\int(Xdx+Ydy)}},$$

the integral being taken between the proper limits, will determine the time of describing any length of path.

The problem—to determine the law of force necessary in order that a particle may describe a given plane curve in a given manner (that is, so that it shall have a given velocity at any point)—may be solved thus:

Let v be the velocity at the point (xy),

$$\frac{dx}{dt}=\frac{dx}{ds}v,\quad\text{and}\quad\frac{d^2x}{dt^2}=v\frac{d}{ds}\left(v\frac{dx}{ds}\right);$$

$$\text{similarly, }\frac{d^2y}{dt^2}=v\frac{d}{ds}\left(v\frac{dy}{ds}\right),$$

which give the component accelerating effects of the force parallel respectively to the axes of x and y. But perhaps the simplest way of solving the problem will be to find the tangential $\left(v\frac{dv}{ds}\right)$ and normal $\left(\frac{v^2}{\rho}\right)$ component accelerating effects of the force.

The differential equation of the path of a particle acted

on by forces parallel to a plane, if the particle move in the plane, or of the projection of the path on the plane, if the forces be independent of the distance of the particle from the plane, may be expressed in terms of polar co-ordinates.

Let r, θ be the polar co-ordinates of the particle in the plane, or of its projection, as the case may be; let the forces be resolved along and perpendicular to r, and let R and T be their accelerating effects in these directions. The equations of motion are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = R, \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = T.$$

From the second of these we have

$$r^2\frac{d\theta}{dt}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = Tr^3\frac{d\theta}{dt};$$

or, if h be put for $r^2\dfrac{d\theta}{dt}$ and u for $\dfrac{1}{r}$,

$$h\frac{dh}{d\theta} = \frac{T}{u^3};$$

$$\therefore h^2 = 2\int\frac{T\,d\theta}{u^3},$$

in which the integral is supposed to include the requisite constant.

$$\text{Again,} \quad \frac{dr}{dt} = \frac{d}{d\theta}\left(\frac{1}{u}\right)\frac{d\theta}{dt} = -h\frac{du}{d\theta},$$

$$\frac{d^2r}{dt^2} = -hu^2\frac{d}{d\theta}\left(h\frac{du}{d\theta}\right) = -h\frac{dh}{d\theta}u^2\frac{du}{d\theta} - h^2u^2\frac{d^2u}{d\theta^2}$$

$$= -\frac{T\dfrac{du}{d\theta}}{u} - h^2u^2\frac{d^2u}{d\theta^2}.$$

Hence, substituting in the first equation,

$$\frac{T\dfrac{du}{d\theta}}{u} + h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3 + R = 0.$$

From which, by putting for h^2 its value, we get

$$\frac{d^2u}{d\theta^2} + u + \frac{R + \dfrac{Tdu}{ud\theta}}{2u^2 \displaystyle\int \frac{Td\theta}{u^3}} = 0,$$

the equation sought.

To find the time of describing any portion of the path,

$$r^2 \frac{d\theta}{dt} = h;$$

$$\therefore t = \int \frac{d\theta}{hu^2} = \int \frac{d\theta}{u^2 \left(2 \displaystyle\int \frac{Td\theta}{u^3}\right)^{\frac{1}{2}}}$$

the integral being taken between the proper limits.

V. *The motion of a particle acted on by a force whose line of action always intersects a fixed straight line.*

38. Let the fixed straight line be taken for axis of z, and a plane perpendicular to it for plane of xy. Let x, y, z be the co-ordinates of the particle at the end of the time t. The force on the particle may be resolved in directions parallel and perpendicular to the axis of z; let Z and R be its component accelerating effects in these directions respectively. The component R may be further resolved into $R\dfrac{x}{\sqrt{x^2+y^2}}$ and $R\dfrac{y}{\sqrt{x^2+y^2}}$ parallel respectively to the axes of x and y. Hence the equations of motion are

$$\frac{d^2x}{dt^2} = \frac{Rx}{\sqrt{x^2+y^2}}, \quad \frac{d^2y}{dt^2} = \frac{Ry}{\sqrt{x^2+y^2}}, \quad \frac{d^2z}{dt^2} = Z.$$

The equations may be otherwise written; let r be the perpendicular distance of the particle from the fixed line, θ the angle which r makes with a fixed plane passing through the line, and z the distance of the particle from a fixed plane

perpendicular to the line. We have, by resolving accelerations along and perpendicular to r,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = R, \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0, \quad \frac{d^2z}{dt^2} = Z.$$

From the second of these equations

$$r^2\frac{d\theta}{dt} = \text{constant throughout the motion,}$$

therefore the area swept out by the projection of r on a plane perpendicular to the fixed line is proportional to the time in which it is swept out.

The projection of the path on a plane perpendicular to the fixed line has manifestly the properties of a central orbit, in which the center of force corresponds to the point where the line pierces the plane.

If R do not involve z, the differential equation of the projection of the path is found to be (as in art. 28)

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} + \frac{Rr^2}{h^2} = 0,$$

h being the constant value of $r^2\dfrac{d\theta}{dt}$.

$$\text{Since } \frac{d^2z}{dt^2} = \frac{h}{r^2}\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dz}{d\theta}\right), \text{ therefore}$$

$$\frac{h^2}{r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dz}{d\theta}\right) = Z,$$

which may be used to find z in terms of r and θ. These last two equations may be employed to find the law of the force, intersecting a fixed line, necessary for the description of a given path.

Again, since

$$2\frac{dr}{dt}\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\} = 2\frac{dr}{dt}\left(\frac{d^2r}{dt^2} - \frac{h^2}{r^3}\right)$$

$$= \frac{d}{dt}\left\{\left(\frac{dr}{dt}\right)^2 + \frac{h^2}{r^2}\right\} = \frac{d}{dt}\left\{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right\};$$

hence by multiplying the first equation of motion by $2\frac{dr}{dt}$, the third by $2\frac{dz}{dt}$, and integrating,

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 2\int(R\,dr + Z\,dz),$$

which gives the velocity of the particle if $Rdr + Zdz$ be a perfect differential.

VI. *The motion of a particle acted on by any force whatever.*

39. Let x, y, z be the co-ordinates of the particle at the end of the time t measured from a fixed epoch, and let X, Y, Z be the component accelerating effects of the force acting on it in directions of the co-ordinate axes. The equations of motion are

$$\frac{d^2x}{dt^2} = X, \qquad \frac{d^2y}{dt^2} = Y, \qquad \frac{d^2z}{dt^2} = Z.$$

Multiplying these equations in order by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$, $2\frac{dz}{dt}$ and integrating, we have

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 2\int(X\,dx + Y\,dy + Z\,dz),$$

which gives the velocity if the quantity under the integral sign be a perfect differential. The integral is supposed to contain the requisite constant.

In order to determine the path it will be necessary to obtain from the equations of motion (by help of the last equation if necessary) two integral equations in which t does not appear. These will represent two surfaces whose line of intersection is the path of the particle. When the path is known, the time of describing any length of the path is

$$= \int \frac{ds}{\sqrt{2\int(X\,dx + Y\,dy + Z\,dz)}}$$

taken between the proper limits, ds being an element of the path's length.

To find the law of force necessary in order that a particle may describe a given path in a given manner, let v denote the velocity of the particle at the point (x, y, z). Since

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{ds}\cdot\frac{ds}{dt}\right) = v\frac{d}{ds}\left(v\frac{dx}{ds}\right),$$

and similar expressions hold for $\dfrac{d^2y}{dt^2}$ and $\dfrac{d^2z}{dt^2}$, therefore

$$X = v\frac{d}{ds}\left(v\frac{dx}{ds}\right),\quad Y = v\frac{d}{ds}\left(v\frac{dy}{ds}\right),\quad Z = v\frac{d}{ds}\left(v\frac{dz}{ds}\right)$$

which give the component accelerating effects of the force at any point of the path. The force may also be found from the expressions $v\dfrac{dv}{ds}$ and $\dfrac{v^2}{\rho}$, which denote its component accelerating effects in directions of the tangent and radius of curvature.

The equations of motion may also be written in terms of the polar co-ordinates of the particle. Let r be the distance of the particle from the origin O, θ the angle which r makes with Oz, and ϕ the angle which the plane zOr makes with the plane zOx. Let the component accelerating effects of the force be denoted by R along r, S perpendicular to r in the plane zOr, and T perpendicular to the plane zOr. By cor. 3 of art. 8, we have

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 = R,$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - r\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2 = S,$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right)\sin\theta + 2r\frac{d\theta}{dt}\frac{d\phi}{dt}\cos\theta = T.$$

COR. *Similar orbits may be described by particles under the action of forces having equal intensities and similar directions at similarly situated points; and the velocities of the particles at similarly situated points, and the times of describing similar arcs will be in the sub-duplicate ratio of the lineal dimensions of the orbits.*

Let two particles describe similar orbits whose lineal dimensions are in the ratio of $k : 1$. Let v, v' denote the velocities, and ρ, ρ' the radii of absolute curvature at corresponding points situated at the extremities of similar arcs s, s' measured from similarly situated fixed points in the orbits. The accelerations of motion along ρ and ρ' will be equal if

$$\frac{v^2}{\rho} = \frac{v'^2}{\rho'},$$

or, since $\rho = k\rho'$, if $v^2 = kv'^2$. Let this relation hold; therefore we have, by differentiation,

$$v\frac{dv}{ds} = kv'\frac{dv'}{ds} = kv'\frac{dv'}{ds'}\frac{ds'}{ds}$$

$$= v'\frac{dv'}{ds'},$$

since $s = ks'$ and therefore $1 = k\dfrac{ds'}{ds}$. Hence the accelerations of motion along the tangent are also equal. Consequently the orbits may be described by forces which have equal intensities and similar directions at corresponding points.

To compare the intervals of time (t, t' suppose) in which similar arcs s, s' are described, we have

$$\frac{ds}{dt} = v, \quad \frac{ds'}{dt'} = v';$$

$$\therefore\ dt = \frac{ds}{v} = \frac{kds'}{\sqrt{k}\,v'} = \sqrt{k}\,dt'.$$

Hence, by integration,

$$t = \sqrt{k}\,t',$$

no constant being added since t, t' are supposed to begin with s, s'. It has been shewn that $v = \sqrt{k}v'$; therefore the times of describing similar arcs, and the velocities at corresponding points, are proportional to the square roots of the lineal dimensions of the orbits. (see Newton's *Principia*, Book I. Prop. 58).

CHAPTER IV.

THE FREE MOTION OF TWO MATERIAL PARTICLES ACTING ON ONE ANOTHER.

---

40. Hitherto a moving point has been considered as changing its position only with respect to objects fixed in space; it may likewise be considered as changing its position relative to objects which are not fixed, but which are themselves continually undergoing a change of position. If the co-ordinate axes to which the moving point is referred be moving in space, the co-ordinates of the point will undergo changes due partly to the motion of the point and partly to the motion of the axes; the curve indicated by the successive values of the co-ordinates is called the path of the point relative to these axes, and is the path which the point would actually appear to describe to a spectator partaking of the motion of the co-ordinate axes and consequently regarding them as fixed objects of reference. The simplest case of motion relative to a system of co-ordinate axes is that in which the axes are always parallel to their original directions. If two points be in motion and if through one of them co-ordinate axes be drawn always parallel to fixed directions, the path of the other relative to these axes is called its path relative to the first, or its orbit about the first; hence if x, y, z and x', y', z' be the co-ordinates of the points at any instant relative to co-ordinate axes fixed in space, $x - x'$, $y - y'$, $z - z'$ are the co-ordinates of the first point relative to, or in its orbit about, the second, and $x' - x$, $y' - y$, $z' - z$ are the co-ordinates of the second relative to the first. The velocity with which one point moves in its orbit relative to the other, and the acceleration of motion in the same orbit, are called respectively the relative velocity and relative acceleration of motion of the first with regard to the second.

Two points moving in any manner whatever describe similar figures about one another and about a third point, always taken in the line joining them so as to divide the distance between them into segments which are in a constant ratio.

Since the distance between the points is divided by the third point into segments which are always to one another in the same ratio, therefore each of the segments is to the whole distance in a ratio which is constant throughout the motion; moreover, the distance between the points and the segments into which it is divided are parallel radii-vectors in the orbits described by the points about one another and about the third point. Consequently these orbits are all similar figures.

It is clear that the proposition will hold true if the third point be taken in the production of the line joining the two points, provided only its distances from them be always in a constant ratio. The orbits described by the two points about one another are manifestly equal, similar, and oppositely situated; and the orbits described by them about the third are similar, and oppositely or similarly situated, according as the third point lies in the line joining them or in that line produced.

COR. *Two material particles moving in any manner describe similar figures about one another and about their center of gravity.*

For the center of gravity of two particles divides the distance between them into segments that are inversely as the masses of the particles to which they are adjacent, and the ratio of the masses is a constant ratio.

These properties are evidently geometrical properties arising out of the definition of relative motion, and entirely independent of mechanical considerations. They are equally true for two points moving in any conceivable manner, and for two particles moving according to mechanical laws. The property enunciated in the following proposition is of a like character.

If two points move in any manner, the relative motion of the first with respect to the second is the same as its absolute

motion in space would be if there were combined with its actual motion—initially, a velocity equal and opposite to the initial velocity of the second, and continually throughout the motion, an acceleration of motion equal and opposite to the acceleration of motion of the second.

For by applying equal velocities or equal accelerations to the points, their relative motions are unaffected; but by applying to the second as well as to the first the quantities which the proposition conceives applied to the first only, the second would be reduced to rest, and the motion of the first relative to it would be the same as the absolute motion of the first in space.

41. *If two particles act on one another, the accelerating effect of the force exerted by the first on the second is to the accelerating effect of that exerted by the second on the first as the mass of the first is to the mass of the second.*

This is an immediate consequence of the third law of motion. Let m, m' be the masses of the particles, and let f, f' be the accelerating effects produced respectively by the force which m exerts on m', and by that which m' exerts on m, at any instant. The moving effects of these forces are respectively $m'f$ and mf'; now by the third law of motion the moving effects are equal and opposite,

$$\therefore\ m'f = mf';$$

$$\text{and}\ \therefore\ \frac{f}{f'} = \frac{m}{m'};$$

which proves the proposition.

Hence, if $m\phi(r)$ denote the accelerating effect of the force which a particle of mass m exerts on another of mass m' at a distance r from it, $m'\phi(r)$ denotes the accelerating effect in the opposite direction of the force which m' exerts on m, and $mm'\phi(r)$ denotes the common moving effect of the force which each exerts on the other.

COR. *Assuming as a physical fact that the action of a particle produces the same accelerating effect on all particles (whatever be their masses) which are at the same distance; it follows that the accelerating effects of the actions of different particles on*

any particles at the same distances from them are proportional to the masses of the acting particles.

For if f be the accelerating effect of the action of a particle m on each of the particles m', m'' when placed equidistant from m, and if the actions of m', m'' produce accelerating effects f', f'' on m at the same distance; we have, by the proposition,

$$\frac{f}{f'} = \frac{m}{m'}, \qquad \frac{f}{f''} = \frac{m}{m''};$$

$$\therefore \frac{f'}{f''} = \frac{m'}{m''}.$$

Hence the masses of particles may be compared and measured by the accelerating effects which their actions produce on particles placed at a given distance from them. Thus if a particle attract according to the law of the inverse square of the distance, the accelerating effect of its attraction on a particle at distance r may be represented by $\frac{m}{r^2}$, m being the mass of the particle measured by the acceleration of motion which its attraction produces in a particle placed at a distance from it equal to unity.

The masses of the sun and planets (considered as particles) are usually measured by the accelerating effects of their attractions at the unit of distance. But, for the sake of convenience, instead of the units of time and distance being first fixed on arbitrarily, and then the mass of a particle whose attraction at the unit of distance produces an unit of acceleration in an unit of time being defined as the unit of mass, a different order is followed in choosing the units. The mass of the sun is taken for unit of mass; then the unit of time being agreed on, the unit of distance becomes fixed and may be defined to be the distance such that if a particle were to move from rest under the action of a constant force whose intensity is equal to the intensity of the sun's attraction on the particle at that distance, and if at the end of an unit of time the action of the force were to cease and the particle were allowed to move uniformly with the velocity it had ac-

quired, the space which the particle during its uniform motion would describe in an unit of time would be that distance.

42. *To find the equations of motion of two particles which are subject only to their own mutual actions.*

At the end of the time t measured from a certain epoch, let x, y, z and x', y', z' be the co-ordinates, referred to axes fixed in space, of two particles whose masses are m and m'; and let $r_{\prime}$ be the distance between them. If F denote the moving effect of the force with which m attracts m', it will also denote the moving effect of the attraction of m' on m, by the law of action and reaction. The equations of motion of m are therefore

$$m\frac{d^2x}{dt^2} = -F\frac{x-x'}{r_{\prime}}, \quad m\frac{d^2y}{dt^2} = -F\frac{y-y'}{r_{\prime}}, \quad m\frac{d^2z}{dt^2} = -F\frac{z-z'}{r_{\prime}};$$

and those of m' are

$$m'\frac{d^2x'}{dt^2} = F\frac{x-x'}{r_{\prime}}, \quad m'\frac{d^2y'}{dt^2} = F\frac{y-y'}{r_{\prime}}, \quad m'\frac{d^2z'}{dt^2} = F\frac{z-z'}{r_{\prime}};$$

from these equations all the properties of the motion may be found.

COR. The equations of relative motion of each particle with respect to the other are at once found from these equations; those of m relative to m' are evidently

$$m\frac{d^2}{dt^2}(x-x') = -\frac{m+m'}{m'}F\frac{x-x'}{r_{\prime}},$$

$$m\frac{d^2}{dt^2}(y-y') = -\frac{m+m'}{m'}F\frac{y-y'}{r_{\prime}},$$

$$m\frac{d^2}{dt^2}(z-z') = -\frac{m+m'}{m'}F\frac{z-z'}{r_{\prime}}.$$

These equations are manifestly the equations we should have obtained had m' been fixed and had m, besides being subject to the action of m', been acted on by a force whose accelerating effect is equal and opposite to the accelerating effect of the

action of m on m': this might have been immediately concluded from the last proposition of art. 40.

We have

$$\frac{\frac{d^2}{dt^2}(x-x')}{x-x'} = \frac{\frac{d^2}{dt^2}(y-y')}{y-y'} = \frac{\frac{d^2}{dt^2}(z-z')}{z-z'};$$

and by integrating these equations two and two, we get

$$(y-y')\frac{d}{dt}(z-z') - (z-z')\frac{d}{dt}(y-y') = A,$$

$$(z-z')\frac{d}{dt}(x-x') - (x-x')\frac{d}{dt}(z-z') = B,$$

$$(x-x')\frac{d}{dt}(y-y') - (y-y')\frac{d}{dt}(x-x') = C;$$

A, B, C being quantities independent of the time, and therefore depending only on the initial relative positions and relative velocities of the particles. Multiplying these equations in order by $x - x'$, $y - y'$, $z - z'$, and adding,

$$A(x-x') + B(y-y') + C(z-z') = 0.$$

This result shews that the particles may be conceived to be always in a plane whose directional cosines are proportional to the constant quantities A, B, C. Hence a plane may be supposed to move so as always to be parallel to its initial position and always to pass through the particles.

43. *If two particles be subject only to their mutual actions, their center of gravity either is at rest or moves in a straight line with a constant velocity.*

By adding together the equations of motion two and two, we have

$$m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} = 0,\quad m\frac{d^2y}{dt^2} + m'\frac{d^2y'}{dt^2} = 0,$$

$$m\frac{d^2z}{dt^2} + m'\frac{d^2z'}{dt^2} = 0.$$

Now if $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates of the center of gravity at the end of the time t, $(m + m')\bar{x} = mx + m'x'$;

$$\therefore \ (m + m')\frac{d^2\bar{x}}{dt^2} = m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2},$$

and similar expressions hold for $\bar{y}$ and $\bar{z}$;

$$\therefore \ \frac{d^2\bar{x}}{dt^2} = 0, \quad \frac{d^2\bar{y}}{dt^2} = 0, \quad \frac{d^2\bar{z}}{dt^2} = 0.$$

From which we get, by integration

$$\bar{x} = \bar{u}t + a, \quad \bar{y} = \bar{v}t + b, \quad \bar{z} = \bar{w}t + c,$$

$\bar{u}, \bar{v}, \bar{w}, a, b, c$ being quantities independent of the time. Hence the center of gravity moves with the constant velocity $\sqrt{\bar{u}^2 + \bar{v}^2 + \bar{w}^2}$, in the straight line whose equations are

$$\frac{\bar{x} - a}{\bar{u}} = \frac{\bar{y} - b}{\bar{v}} = \frac{\bar{z} - c}{\bar{w}},$$

if $\bar{u}, \bar{v}, \bar{w}$ be not all zero; or it rests at the point (a, b, c) if $\bar{u}, \bar{v}, \bar{w}$ are zero.

If u, v, w and u', v', w' be the initial component velocities of the particles m and m',

$$\bar{u} = \frac{mu + m'u'}{m + m'}, \quad \bar{v} = \frac{mv + m'v'}{m + m'}, \quad \bar{w} = \frac{mw + m'w'}{m + m'}.$$

COR. 1. The equations of relative motion of the particles about their center of gravity are easily found. Let r, r' be the distances of m, m' from their center of gravity. Since $\frac{d^2\bar{x}}{dt^2} = 0$, and $\frac{x - x'}{r_{\prime}} = \frac{x - \bar{x}}{r}$, and similar expressions hold for the other co-ordinates, therefore the equations for m are

$$m\frac{d^2}{dt^2}(x - \bar{x}) = -F\frac{x - \bar{x}}{r}, \quad m\frac{d^2}{dt^2}(y - \bar{y}) = -F\frac{y - \bar{y}}{r},$$

$$m\frac{d^2}{dt^2}(z - \bar{z}) = -F\frac{z - \bar{z}}{r},$$

and similarly, those for m' are

$$m'\frac{d^2}{dt^2}(x' - \bar{x}) = -F\frac{x' - \bar{x}}{r'}, \quad m'\frac{d^2}{dt^2}(y' - \bar{y}) = -F\frac{y' - \bar{y}}{r'},$$

$$m'\frac{d^2}{dt^2}(z' - \bar{z}) = -F\frac{z' - \bar{z}}{r'}.$$

These equations are precisely the same as if the center of gravity had been a fixed point, and the motion had been referred to co-ordinate axes drawn through it.

It is easy to shew from these equations, as in the last article, that planes may be supposed to move parallel to themselves so as to contain respectively each of the particles and their center of gravity, and that these planes are coincident with one another and with the one found in the last article. Since this plane contains the center of gravity, and since the center of gravity moves uniformly, therefore through the center of gravity there may be drawn parallel to fixed directions two straight lines which, throughout the motion, are always parallel to the same directions, move with a constant velocity, and have always the particles in their plane. It is evident that the particles will describe paths in the plane of these lines relative to them; and hence the motions of the particles in space may be supposed to arise from their motions in this plane while at the same time the plane is itself in motion. The plane here spoken of which moves uniformly, continues parallel to its initial position, and always passes through the particles, may be called the *relative plane of motion* of the particles.

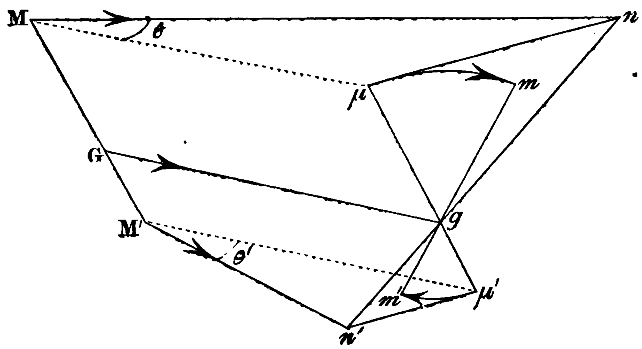
COR. 2. *The equations of motion of the particles in their relative plane of motion are the same as if the plane were at rest.*

For the equations of motion of either particle relative to the center of gravity are the same as if the center of gravity were at rest. In fact if x, y be the co-ordinates of m relative to the rectangular axes drawn in the relative plane of motion through the center of gravity, we have (by the equations of last Cor. if the plane of xy be supposed parallel to the relative plane of motion)

$$m\frac{d^2x}{dt^2} = -F\frac{x}{r}, \quad m\frac{d^2y}{dt^2} = -F\frac{y}{r}.$$

44. *To determine the motions of two particles which are subject only to their own actions on one another.*

The position of the relative plane of motion of the particles may be found at any instant during the motion. For since this position depends entirely on the initial circumstances of the motion, it will be the same as if the particles did not act on one another; but if they did not act on one another, each of them would move uniformly along its initial line of motion with a velocity equal to its initial velocity. Hence, if M, M' be the initial positions of the particles m, m',



G the initial position of their center of gravity, and Mn, $M'n'$ their initial lines of motion, take Mn, $M'n'$ equal to the spaces which the particles would describe in the time t if they moved uniformly along Mn, $M'n'$ with their initial velocities, find g the center of gravity of the particles when at n, n', and through g draw $\mu g \mu'$ parallel to MGM'; the plane of the lines ngn', $\mu g \mu'$ is the relative plane of motion of the particles at the end of the time t. If Gg be joined, and $M\mu$, $M'\mu'$ be drawn parallel to Gg, the straight lines $M\mu$, $M'\mu'$ will be the paths described in the time t by the points of the relative plane of motion which initially coincided with the respective particles, and Gg will be the path described in the same time by the center of gravity. While the relative plane of motion moves from its initial position to its position at the end of the time t, the particles will describe in it paths (μm, $\mu' m'$ in the figure) about g as a fixed point, and μn, $\mu' n'$ will manifestly be tangents to these paths at the points μ, μ' respectively. The absolute paths of the particles in space will be curves, the one passing through

M and m, and the other through M' and m'; the straight lines Mn, $M'n'$ will be tangents to these curves at the points M, M'.

The construction here given for the relative plane of motion will fail if the line ngn' be parallel to MGM'. In this case it is manifest that a straight line may be supposed to move so as always to be parallel to its initial position and to contain the particles; this line may be called the relative line of motion; its position at any instant is determined by the above construction. When a relative line of motion exists, a plane drawn through it parallel to any fixed plane will manifestly be a relative plane of motion.

Having thus determined the position at any instant of the relative plane of motion (or of the relative line of motion when such a line exists), it now remains only to determine the motions of the particles in this plane (or line). Let the center of gravity of the particles, which is a point fixed in the relative plane of motion, be taken for pole. To find the motion of the particle m in the relative plane, take $g\mu$ for prime radius, and denote by r, θ the line gm and the angle μgm, the polar co-ordinates of m at the end of the time t; since (by cor. 2 of last art.) the motion in the relative plane is the same as if that plane were fixed, the equations for m are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{F}{m}, \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0,$$

F being the moving effect of the action of m' on m.

Similarly if, to find the motion of m' in the relative plane, $g\mu'$ be taken for prime radius, and r', θ be taken to denote respectively the line gm' and the angle $\mu'gm'$, we have

$$\frac{d^2r'}{dt^2} - r'\left(\frac{d\theta}{dt}\right)^2 = -\frac{F}{m'}, \quad \frac{1}{r'}\frac{d}{dt}\left(r'^2\frac{d\theta}{dt}\right) = 0.$$

If the particles have a relative line of motion, calling r and r' the distances of the particles along the relative line from the center of gravity, we have in the same way

$$\frac{d^2r}{dt^2} = -\frac{F}{m}, \quad \frac{d^2r'}{dt^2} = -\frac{F}{m'},$$

for the equations of motion of the particles in the relative line of motion.

The moving effect F of the force which each of the particles exerts on the other is usually given in terms of the distance between the particles; but it may be at once expressed in terms of r or r', for since $mr = m'r'$,

$$\therefore \frac{r}{m'} = \frac{r'}{m} = \frac{r+r'}{m+m'}.$$

The equations of motion of either particle in its orbit about the other, in the relative plane of motion, are found by adding together the above equations of motion of the particles about their center of gravity. We thus get

$$\frac{d^2}{dt^2}(r+r') - (r+r')\left(\frac{d\theta}{dt}\right)^2 = -\frac{m+m'}{mm'}F, \quad \frac{1}{r+r'}\frac{d}{dt}\left\{(r+r')^2\frac{d\theta}{dt}\right\} = 0.$$

From the foregoing equations everything connected with the motions may be found. The forms of the equations shew that the motions of the particles in the relative plane of motion are of a kind similar to those treated of in arts. 25—35, or in arts. 15—20.

COR. 1. *The relative orbits of the particles about each other and about their center of gravity are plane curves; but their absolute paths in space are plane curves only when their initial lines of motion are in one plane.*

For the particles are always in the relative plane of motion, and this plane moves in the plane of its initial position only when the lines of initial motion are in one plane.

COR. 2. Let v, v' be the initial velocities of the particles, i the inclination to one another of the initial lines of motion, and $\bar{v}$ the velocity of the center of gravity. Through M draw a straight line parallel and equal to $M'n'$, and join the extremity of this line by a straight line with n; if the joining line be divided into segments in the ratio of m' to m, the velocities of increase of these segments will manifestly be the initial velocities of the particles in the relative plane of motion, and the velocity of increase of a line joining M with

the point of section will be the velocity of the center of gravity. Hence the initial velocities of m and m' in the relative plane of motion, that is, along the lines μn and $\mu' n'$, are respectively

$$\frac{m'\sqrt{v^2 + v'^2 - 2vv'\cos i}}{m + m'}, \quad \frac{m\sqrt{v^2 + v'^2 - 2vv'\cos i}}{m + m'}$$

$$\text{and } \bar{v} = \frac{1}{m + m'}\sqrt{m^2 v^2 + m'^2 v'^2 + 2mm'vv'\cos i}.$$

Let the line of motion of the center of gravity make with the initial lines of motion angles θ, θ' (the angles $\mu M n$, $\mu' M' n'$ in the figure); we have

$$\frac{\sin\theta}{\sin\theta'} = \frac{\sin\theta}{\sin\mu}\,\frac{\sin\mu'}{\sin\theta'} = \frac{\mu n}{Mn}\,\frac{M'n'}{\mu' n'} = \frac{m'v'}{mv},$$

$$\text{since } \frac{n\mu}{n'\mu'} = \frac{ng}{n'g} = \frac{m'}{m}. \quad \text{Also we have } \theta + \theta' = i;$$

$$\therefore \sin\theta = \frac{m'v'\sin i}{(m + m')\bar{v}}, \quad \sin\theta' = \frac{mv\sin i}{(m + m')\bar{v}}.$$

Again, to find the angle (β suppose) which the line of motion of the center of gravity makes with the relative plane of motion, let α, α' be the angles which the lines of initial motion make with the line which passes through the initial positions of the particles, and let planes drawn through this last line and through the respective initial lines of motion make an angle ϕ with one another. If the figure be projected on a plane perpendicular to MM', the projection of the relative plane of motion will be a straight line (since this plane is parallel to MM' and therefore perpendicular to the plane of projection); the projections of n, n' will move along lines inclined to one another at an angle ϕ with velocities equal respectively to $v \sin \alpha$, $v' \sin \alpha'$, and the distance between these projections will increase with a velocity

$$= \sqrt{v^2\sin^2\alpha + v'^2\sin^2\alpha' - 2vv'\sin\alpha\sin\alpha'\cos\phi};$$

moreover, the projection of the relative plane will move with a velocity in direction perpendicular to itself equal to $\bar{v} \sin \beta$.

Hence $vv' \sin \alpha \sin \alpha' \sin \phi$

$$= \bar{v} \sin \beta \sqrt{v^2 \sin^2 \alpha + v'^2 \sin^2 \alpha' - 2vv' \sin \alpha \sin \alpha' \cos \phi}\,;$$

$$\sin \beta = \frac{vv' \sin \alpha \sin \alpha' \sin \phi}{\bar{v} \sqrt{v^2 \sin^2 \alpha + v'^2 \sin^2 \alpha' - 2vv' \sin \alpha \sin \alpha' \cos \phi}}.$$

Of course ϕ may be expressed in terms of α, α', and i, by help of the formula

$$\cos i = \cos \alpha \cos \alpha' + \sin \alpha \sin \alpha' \cos \phi.$$

From the foregoing formulæ we have the following consequences, which however are themselves apparent. The center of gravity will be at rest only when

$$m^2 v^2 + m'^2 v'^2 + 2mm'vv' \cos i = 0,$$

that is, either when the initial velocities are both zero, or when the initial lines of motion are parallel (or coincident) and the initial velocities are inversely as the masses of the particles and in opposite directions.

The relative plane of motion will move in the plane of its initial position when

$$\sin \alpha \sin \alpha' \sin \phi = 0\,;$$

that is, when the initial lines of motion lie in one plane. Under this case is included every case in which there exists a relative line of motion; in every such case it is clear that β must be indeterminate and therefore

$$v^2 \sin^2 \alpha + v'^2 \sin^2 \alpha' - 2vv' \sin \alpha \sin \alpha' \cos \phi = 0\,;$$

or if $\phi = 0$

$$(v \sin \alpha - v' \sin \alpha')^2 = 0.$$

Consequently, either the initial lines of motion coincide with the line passing through the initial positions of the particles, the initial velocities being any whatever; or the initial lines of motion are parallel, and the initial velocities equal and in the same direction; or the initial lines of motion intersect, and the initial velocities are either both to or both from the point of intersection and proportional to the distances of the initial positions of the particles from the point of intersection.

45. *To find the motions of two particles each of which attracts with a force whose eccelerating effect at any distance is inversely proportional to the square of that distance.*

Let m and m' be the masses of the particles measured by the accelerating effects of their attractions at the unit of distance, and let $r_{,}$ be the distance between them at the end of the time t. The accelerating effect of the attraction which m exerts on m' is $\frac{m}{r_{,}^2}$, and the accelerating effect of the attraction which m' exerts on m is $\frac{m'}{r_{,}^2}$. If r, r' be the segments adjacent to m, m' respectively, into which $r_{,}$ is divided by the center of gravity, we have,

$$\frac{m}{r_{,}^2} = m\left\{\frac{m}{(m+m')\,r'}\right\}^2 = \frac{m^3}{(m+m')^2 r'^2};$$

$$\text{and similarly, } \frac{m'}{r_{,}^2} = \frac{m'^3}{(m+m')^2 r^2}.$$

Let the position of the relative plane of motion at the end of the time t be found as in last article from the initial circumstances of the motion. The motion of m in this plane, relative to the center of gravity is then known from the equations

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{m'^3}{(m+m')^2 r^2}, \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0.$$

The equations of motion of m' in the relative plane are

$$\frac{d^2 r'}{dt^2} - r'\left(\frac{d\theta}{dt}\right)^2 = -\frac{m^3}{(m+m')^2 r'^2}, \quad \frac{1}{r'}\frac{d}{dt}\left(r'^2\frac{d\theta}{dt}\right) = 0.$$

Hence, (art. 28) the differential equations of the orbits of m and m' in the relative plane of motion are respectively

$$\frac{d^2 u}{d\theta^2} + u - \frac{m'^3}{h^2(m+m')^2} = 0, \quad \frac{d^2 u'}{d\theta^2} + u' - \frac{m^3}{h'^2(m+m')^2} = 0;$$

in which u and u' are written for the reciprocals of r and r', and h and h' are quantities depending on the initial circumstances and evidently such that $m^2 h = m'^2 h'$. The initial mo-

tions of the particles in the relative plane are to be found by the formulæ of cor. 2 of last article.

By integrating these equations as in art. 34, the orbits of the particles in the relative plane of motion will be determined. Each of the orbits will be a conic section having the center of gravity in one focus.

If the initial circumstances be such that there is a relative line of motion, the motions of the particles in this line are to be found from the equations

$$\frac{d^2 r}{dt^2} = -\frac{m'^3}{(m+m')^2 r^2}, \quad \frac{d^2 r'}{dt^2} = -\frac{m^3}{(m+m')r'^2};$$

which are to be integrated as in art. 19.

Cor. Let the orbits which the particles describe about one another and about their center of gravity be ellipses, and let a be the semi-axis major of the ellipse which each describes about the other. The semi-axis major of the ellipse described by m in the relative plane of motion will therefore be

$$= \frac{am'}{m+m'}.$$

Hence (cor. art. 31) the periodic time of each particle in its orbit about the other is

$$= 2\pi\sqrt{\left(\frac{am'}{m+m'}\right)^3 \frac{(m+m')^2}{m'^3}} = 2\pi\sqrt{\frac{a^3}{m+m'}}.$$

If one of the particles (m' for instance) be kept fixed in space, the other may describe an elliptic orbit about it as about a fixed center of force; if a' be the semi-axis major in this elliptic orbit, the periodic time is

$$2\pi\sqrt{\frac{a'^3}{m'}}.$$

Let this periodic time be equal to the other just found, and we have

$$\frac{a^3}{m+m'} = \frac{a'^3}{m'};$$

$$\therefore\ a : a' = (m+m')^{\frac{1}{3}} : m'^{\frac{1}{3}},$$

that is, *if two particles describe about one another ellipses, the major-axis of the ellipse which each describes about the other is to the major-axis of the ellipse which one of them may describe in the same periodic time about the other fixed, as the cube root of the sum of the masses of the particles is to the cube root of the mass of the fixed particle.* (NEWTON'S *Principia.* Book I. Prop. 60.)

46. *To find the motions of two particles acting on one another and acted on by given forces.*

Let x, y, z and x', y', z' be the co-ordinates of the particles at the end of the time t elapsed since a fixed epoch, relative to a system of co-ordinate axes fixed in space, m, m' the masses of the particles, and F the moving effect of the force which each exerts on the other; also let X, Y, Z be the component moving effects of the force acting on m, and X', Y', Z' those of the force acting on m'. The equations of motion of m are

$$m\frac{d^2x}{dt^2} = -F\frac{x-x'}{r_{,}} + X,$$

$$m\frac{d^2y}{dt^2} = -F\frac{y-y'}{r_{,}} + Y,$$

$$m\frac{d^2z}{dt^2} = -F\frac{z-z'}{r_{,}} + Z,$$

$r_{,}$ being the distance between the particles; and the equations of motion of m' are

$$m\frac{d^2x'}{dt^2} = -F\frac{x'-x}{r_{,}} + X',$$

$$m'\frac{d^2y'}{dt^2} = -F\frac{y'-y}{r_{,}} + Y',$$

$$m'\frac{d^2z'}{dt^2} = -F\frac{z'-z}{r_{,}} + Z'.$$

From these equations everything connected with the motion is to be found; if the forces be functional of the positions of the particles, the equations are six for determin-

ing the six quantities x, y, z, x', y', z' in terms of the time t; or if the paths of the particles be given the forces which must act on the particles in order that they may describe these paths in a given manner may be found.

COR. If $\bar{x}$, $\bar{y}$, $\bar{z}$ be the co-ordinates of the center of gravity of the particles at the end of the time t, we have by adding together two and two the equations of motion

$$(m + m')\frac{d^2\bar{x}}{dt^2} = X + X',$$

$$(m + m')\frac{d^2\bar{y}}{dt^2} = Y + Y',$$

$$(m + m')\frac{d^2\bar{z}}{dt^2} = Z + Z'.$$

These are the same as the equations of motion of a particle, whose mass is $m + m'$, acted on by forces whose directions and moving effects are the same as the directions and moving effects of the forces acting on the particles. Hence *the motion of the center of gravity of two particles, which act on one another and which are acted on by given forces, is the same as the motion of a particle, of mass equal to the sum of the masses of the particles, would be were it acted on by forces the same in intensity and direction as the given forces which act on the particles.*

The collision of smooth balls.

47. Although there is reason for supposing that every force requires to act during a finite period of time in order to produce a finite change of velocity in a material particle, yet there are some forces which produce finite changes of velocity in intervals of time so short as to be quite insensible. The forces which are brought into play when two balls are made to strike one another are of this kind; the motions of the balls undergo very perceptible changes while the collision is sensibly instantaneous. In such cases of motion, instead of considering the accelerating effect of the forces brought into play at every instant during the collision, it will be sufficient to consider the whole change of velocity

produced by the forces during the short time of their action, as if the change had been produced instantaneously. Forces which produce finite changes of velocity in insensibly short intervals of time are called impulses, and the changes of momenta which they produce are called their impulsive effects.

As balls are extended bodies they cannot be considered particles. By help of the two following propositions, however, the motions of two balls will become a question of the motions of two particles.

(1) *When a body is acted on by any forces, the motion of its center of gravity is the same as the motion of a particle, whose mass is equal to the mass of the body, would be were it acted on by forces equal and parallel to the forces which act on the body.*

The body may be supposed to be made up of an infinite number of particles acting on one another with forces sufficient to preserve them in the relative positions which they hold in the body; and, since the actions on one another of any two of these particles are mutual, equal, and opposite, therefore the sum of the resolved parts of all the actions of the particles on one another estimated in any direction is zero. But by a method precisely similar to that of the cor. in last article, it appears that the sum of the products of the mass of each particle multiplied into its acceleration of motion in a particular direction, is equal to the sum of the moving effects in the same direction of the actions of the particles on one another and of the forces acting on the body. Hence the sum of the masses of the particles, that is, the mass of the body, multiplied into the acceleration of motion in the given direction of the center of gravity, is equal to the sum of the moving effects, estimated in the same direction, of the forces which act on the body. Consequently the motion of the center of gravity is the same as if the whole mass of the body were condensed into a particle, and this particle were acted on by the forces which act on the body.

COR. *If a body be acted on by no forces, its center of gravity either will be at rest or will move uniformly.*

(2) *When two bodies act on one another, the first exerts on the second a force equal and opposite to the force which the second exerts on the first.*

Each of the bodies may be supposed to be made up of an infinite number of particles, and each particle of one body exerts on each particle of the other body a force equal and opposite to that which the latter particle exerts on the former. Consequently the resultant action of the particles of one body on the particles of the other is equal and opposite to the resultant action of the particles of the latter on those of the former; in other words, the actions of the bodies are mutual, equal, and opposite.

These general propositions are immediately applicable to the case of balls impinging on one another. The first shews that the balls may be considered particles situated at their centers of gravity, and the second shews that the balls act on one another in the same way as if they were particles. The balls therefore will move and their motions may be investigated as if they were particles surrounded by impenetrable but immaterial atmospheres, (at least, impenetrable so far as the bodies of the balls are); if the balls be homogeneous the particles will be at the centers of their atmospheres. In what follows, when the motion or velocity of a ball is spoken of, the motion or velocity of its center of gravity is meant, or of the (imaginary) particle at the center of gravity which moves in the same manner.

Before considering the motions of two smooth balls impinging on one another, it will be necessary to premise another proposition, viz.,

When two smooth surfaces are in contact, the pressure which each surface exerts on the other at the point of contact acts normally to the surfaces at that point.

A surface is said to be smooth when it is incapable of offering resistance to a body which tends to slip along the surface. Now suppose the smooth surfaces of two bodies to be in contact, and suppose, if possible, that the pressure of the first on the second is not in the normal direction; since the second exerts on the first an equal and opposite pressure, therefore this pressure does not act normally; it may consequently be resolved into two component pressures, one of which acts normally and the other acts along the surface; but the latter component would resist any tendency

which the first surface might have to slip along the second, which is inconsistent with the definition of a smooth surface. Hence the mutual pressure between two surfaces in contact cannot act but normally.

If a particle press against a smooth surface, it follows in the same way that the pressure between the particle and the surface acts in direction of the normal to the surface.

48. When two smooth spherical balls strike one another the laws which regulate the impact may be stated as follows:—Let the velocity of each ball be resolved into two component velocities one along and the other perpendicular to the line of collision, that is, the line joining the centers of the balls when their surfaces are in contact;

(1) *The component velocities of the balls perpendicular to the line of collision are unaffected by the impact.*

(2) *The algebraic sum of the component momenta of the balls in direction of the line of collision is the same after as before the impact.*

(3) *The relative velocity of the balls in direction of the line of collision, after the impact, is to their relative velocity in the same direction, before the impact, in a ratio which depends entirely on the substances composing the balls.*

The first of these laws follows from the propositions of last article; for since the surfaces of the balls are smooth the pressure between the balls at any instant during the impact acts wholly along the line of collision, and therefore the component velocities of the balls perpendicular to this line are not changed by the impact.

The second law is likewise a consequence of the same propositions. Let a ball of mass m impinge on another ball of mass m', and at any instant (t) during the impact let v, v' be the component velocities of these balls in direction of the line of collision, and P the moving effect of the force which each exerts on the other; therefore

$$m\frac{dv}{dt} = -P, \quad m'\frac{dv'}{dt} = P.$$

$$\text{Hence } m\frac{dv}{dt} + m'\frac{dv'}{dt} = 0,$$

$$\text{and } mv + m'v' = \text{a constant quantity};$$

that is, the sum of the momenta of the balls along the line of collision is the same at every instant during the collision, and is therefore the same after the collision as before it. It is evident that a component velocity along the line of collision in one direction being considered positive, a component velocity in the opposite direction must be reckoned negative.

The third law is entirely a matter of experiment. Let v, v' be the component velocities, in direction of the line of collision, of the balls m, m' before the impact, and let $v_{,}$, $v_{,}'$ be their component velocities in the same direction after the impact; $v - v'$ is therefore the relative velocity of the balls before impact, and $v_{,}' - v_{,}$ is their relative velocity after; and the law gives

$$\frac{v_{,}' - v_{,}}{v - v'} = e,$$

in which e is a quantity depending only on the materials of the balls; it is usually called the elasticity of the balls. When $e = 0$, the balls are said to be non-elastic, and they move on together after the impact; when $e = 1$ they are said to be perfectly elastic; in the case of no bodies does e lie without the limits 0 and 1.

If the impact be supposed to begin at the time 0 and to end at the time τ, and if P denote the moving effect of the pressure between the balls at an instant t during the impact; we have

$$m(v_{,} - v) = -\int_0^\tau P\,dt, \quad m'(v_{,}' - v') = \int_0^\tau P\,dt;$$

$$\text{also } v_{,}' - v_{,} = e(v - v');$$

$$\therefore \int_0^\tau P\,dt = (1 + e)\frac{mm'(v - v')}{m + m'},$$

which expresses the impulsive effect of the action of the balls on one another. From this it appears that if two balls of elasticity e impinge on one another, the impulsive effect of their action is to the impulsive effect of what their action would have been had they been non-elastic as $1 + e$ is to 1. Hence the rule frequently given for calculating the motions of impinging balls:—Calculate the impulsive action between

the balls as if they were non-elastic, and then calculate the motion of each ball as if it were subject to an action equal to the calculated impulsive action multiplied into $1 + e$.

To calculate the motion of a smooth ball after impact on a fixed plane, let the velocity be resolved into components parallel and perpendicular to the plane; the component velocity parallel to the plane will be unaffected by the impact, and the component velocity perpendicular to the plane after the impact will be to the component velocity perpendicular to the plane before the impact in a ratio which depends only on the materials of the ball and plane. This principle is simply the adaptation of the first and third laws of impact above to the case in which one of the balls is fixed, or rather to the case in which the mass of one of the balls is infinitely great.

49. *When two smooth balls impinge directly on one another, to determine their motions after the collision.*

The impact of balls is called direct or oblique according as the centers of the balls move or do not move in the line of collision.

Let a smooth ball of mass m moving with a velocity v impinge directly on another ball of mass m' moving in the same direction with a velocity v'. Let e be the elasticity of the balls; and let $v_,v_,'$ be their respective velocities estimated in the same direction after the impact.

By the law of constant momenta we have

$$mv_, + m'v_,' = mv + m'v',$$

and by the law of relative velocities

$$v_,' - v_, = e(v - v').$$

From these equations we get

$$v_, = \frac{mv + m'v' - m'e(v - v')}{m + m'}, \quad v_,' = \frac{mv + m'v' + me(v - v')}{m + m'},$$

which determine the motions of the balls after the impact. A motion in the direction opposite to that in which the motion is supposed will be indicated by a negative velocity.

COR. 1. If the balls be non-elastic, by putting $e = 0$, we get the velocity of each ball after the impact

$$= \frac{mv + m'v'}{m + m'}.$$

If the balls be perfectly elastic the velocities of the balls after the impact will be found by putting $e = 1$.

If the ball m moving with velocity v overtake the ball m' moving with velocity v', m will rebound backwards if

$$mv + m'v' \text{ be less than } m'e(v - v').$$

If the ball m moving with velocity v meet the ball m' moving in the opposite direction with velocity v', the velocity of m after the impact in direction of its initial motion is

$$= \frac{mv - m'v' - m'e(v + v')}{m + m'},$$

and the velocity of m' in the same direction

$$= \frac{mv - m'v' + me(v + v')}{m + m'}.$$

In a similar way the formulæ may be adapted to any case of direct impact.

COR. 2. From the equations in the proposition we have

$$m'(v_{\prime}' - v') = m(v - v_{\prime}),$$

$$\text{and } v_{\prime}' + v' = v + v_{\prime} - (1 - e)(v - v').$$

Multiplying these together and transposing, we get

$$mv_{\prime}^2 + m'v_{\prime}'^2 = mv^2 + m'v'^2 - m(1 - e)(v - v')(v - v_{\prime}),$$

$$\text{or since } v - v_{\prime} = \frac{m'(1 + e)(v - v')}{m + m'},$$

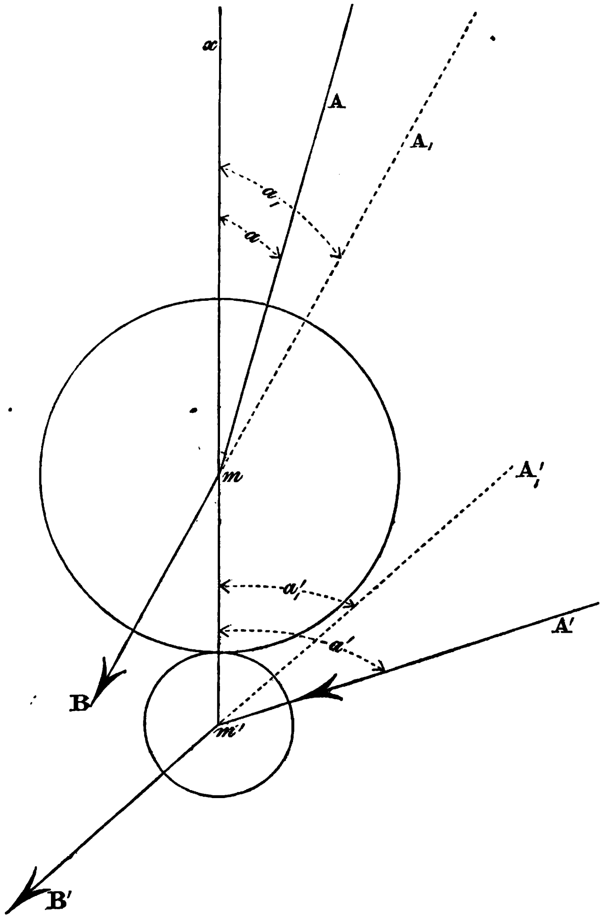
$$mv_{\prime}^2 + m'v_{\prime}'^2 = mv^2 + m'v'^2 - \frac{mm'(1 - e^2)(v - v')^2}{m + m'}.$$

This shews that the sum of the *vires-vivæ* of the balls after the impact is always less than the sum of their *vires-vivæ* before the impact, unless the balls be perfectly elastic. The loss of *vis-viva* sustained in consequence of the impact will be greatest in the case of non-elastic balls.

50. *When two smooth balls impinge in any manner on one another, to determine their motions after the impact.*

It is manifest that the motion of each ball will be in the plane passing through the line of its initial motion and the line of collision.

Let a ball of mass m moving with velocity v along the line Am impinge on the ball of mass m' moving with velocity v' along the line $A'm'$; and after the impact let $v_{,} v_{,}'$ be the velocities of the balls, and $A_{,}mB$, $A_{,}'m'B'$ their lines of motion; let $m'mx$ be the line of collision, and put $Amx = a$, $A'm'x = a'$, $A_{,}mx = a_{,}$, and $A_{,}'m'x = a_{,}'$.



Since the component velocities perpendicular to the line of collision are not changed, therefore

$$v_{,} \sin a_{,} = v \sin a, \qquad v_{,}' \sin a_{,}' = v' \sin a',$$

and by the law of conservation of momenta along the line of collision,

$$mv_{,} \cos a_{,} + m'v_{,}' \cos a_{,}' = mv \cos a + m'v' \cos a',$$

and again, by the law of relative velocities along the line of collision

$$v_{,}' \cos a_{,}' - v_{,} \cos a_{,} = e\,(v \cos a - v' \cos a').$$

From these four equations everything connected with the motion is to be found. From them we get

$$v_{,} = \sqrt{v^2 \sin^2 a + \left\{\frac{mv\cos a + m'v'\cos a' - m'e\,(v\cos a - v'\cos a')}{m + m'}\right\}^2},$$

$$v_{,}' = \sqrt{v'^2 \sin^2 a' + \left\{\frac{mv\cos a + m'v'\cos a' + me\,(v\cos a - v'\cos a')}{m + m'}\right\}^2},$$

$$\tan a_{,} = \frac{(m + m')\, v \sin a}{mv \cos a + m'v' \cos a' - m'e\,(v \cos a - v' \cos a')},$$

$$\tan a_{,}' = \frac{(m + m')\, v' \sin a'}{mv \cos a + m'v' \cos a' + me\,(v \cos a - v' \cos a')}.$$

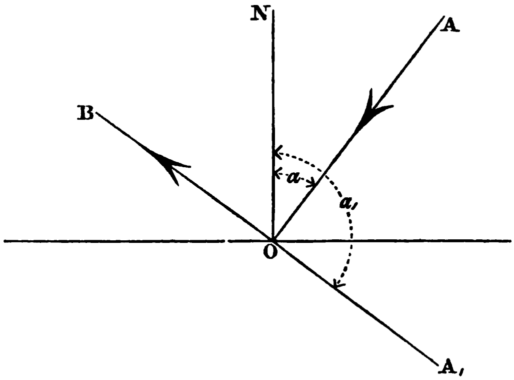
With regard to the algebraic signs of the quantities involved in this question, the simplest way seems to be to consider the velocities as absolutely positive quantities and to let the signs of the angular functions determine the directions of motion; thus in the figure the angles are measured from the same end of the line of collision to the lines which lie in directions opposite to the directions of motion.

When the motions of the balls are in one plane before the impact, the motions will continue in that plane after the impact; if one of the balls be at rest before the impact it will move in the line of collision after.

51. *To determine the motion of a smooth ball after impact against a fixed plane.*

Let $vv_{,}$ be the velocities of the ball before and after the impact, and let $aa_{,}$ be the angles which the directions opposite to the directions of motion make with the normal to the fixed plane. The motion is manifestly in one plane,

viz. the plane, through the initial line of motion, perpen-



dicular to the fixed plane. Since the velocity perpendicular to the normal is unchanged

$$v_{\prime} \sin a_{\prime} = v \sin a,$$

and, by the relation of the relative component velocities,

$$-v_{\prime} \cos a_{\prime} = ev \cos a;$$

$$\therefore\ v_{\prime} = v\sqrt{\sin^2 a + e^2 \cos^2 a}, \quad \tan a_{\prime} = -\frac{1}{e} \tan a.$$

The negative sign of $\tan a_{\prime}$ indicates that $a_{\prime}$ is obtuse, and therefore that the ball rebounds from the plane on the opposite side of the normal.

If the ball and plane be non-elastic, the ball will move along the plane after the impact; if the elasticity be perfect, the lines of motion before and after the impact will make equal angles with the normal.

CHAPTER V.

THE CONSTRAINED MOTION OF PARTICLES.

52. When a particle moves subject to some geometrical condition, such as to move in a tube, or along a surface, or to be always at a certain distance from a point, its motion is said to be constrained. Since the constraints modify the motion, they give rise to forces; the motion of the particle must therefore be determined by introducing the effects of these forces of constraint into the equations of motion, and then proceeding precisely as in a case of free motion.

If a particle be constrained to move along a smooth curve or smooth surface, the force which the curve or surface exerts on the particle at any instant is perpendicular to the direction of motion. For if it were not, it might be resolved into a force perpendicular and another parallel to the direction of motion, of which the latter would produce a resistance to motion along the curve or surface; but this is impossible since perfect smoothness is supposed.

If however the curve or surface along which a particle is constrained to move be rough, the force of constraint will not act in direction perpendicular to the curve or surface. It is usual to suppose that the force of constraint, in such case, acts in direction making with the curve or surface an angle independent of the rate of motion, and depending only on the materials which compose the particle and the curve or surface; in other words, if the force of constraint be resolved into components, one perpendicular and the other parallel to the direction of motion, the latter component acts contrary to the direction of motion, and is to the other component in a ratio depending only on the materials of the particle and the curve or surface. Little however is known on this subject.

In all cases where a particle's motion is constrained, the

particle exerts on the constraining machinery a force equal and opposite to that which the machinery exerts on the particle. This is a consequence of the third law of motion and is easily proved as in the second proposition of art. 47.

The following examples will sufficiently shew the way in which questions of constrained motion are treated. Unless the contrary be stated the curves and surfaces are supposed to be perfectly smooth.

53. *To determine the motion of a particle, constrained to move in a given smooth plane curve, and acted on by given forces in the plane of the curve.*

Let rectangular axes in the plane of the curve be taken for reference, and let x, y be the co-ordinates of the particle at the end of the time t elapsed from a fixed epoch. Let m be the mass of the particle, and let X, Y be the component accelerating effects, parallel to the co-ordinate axes, of the force acting on the particle.

Let R denote the moving effect of the force of constraint exerted by the curve on the particle; since the motion of the particle and the forces acting on it are wholly in the plane of the curve, therefore the force R acts in this plane; and since the curve is smooth the force acts along the normal. Hence the equations of motion are

$$m\frac{d^2x}{dt^2} = mX - R\frac{dy}{ds},$$

$$m\frac{d^2y}{dt^2} = mY + R\frac{dx}{ds},$$

s being an arc of the curve intercepted between (x, y) and a fixed point.

From these equations and the equation of the curve everything connected with the motion may be determined.

Multiplying the first equation by $\frac{dx}{dt}$, the second by $\frac{dy}{dt}$ and adding, we have

$$\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} = X\frac{dx}{dt} + Y\frac{dy}{dt},$$

and multiplying by 2, and integrating relative to t, $\left(\frac{ds}{dt}\right)^2$, or

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\int(Xdx + Ydy),$$

the requisite constant being included under the sign of the integral; this gives the velocity of the particle. The time of describing any arc will be found by another integration,

$$t = \int\frac{ds}{\sqrt{2\int(Xdx + Ydy)}}.$$

Again, the effect of the force of constraint may be found by subtracting the first equation of motion when multiplied by $\frac{dy}{ds}$ from the second multiplied by $\frac{dx}{ds}$; this gives,

$$\text{since } \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{ds}\,\frac{ds}{dt}\right) = \frac{d^2x}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dx}{ds}\,\frac{d^2s}{dt^2},$$

$$\text{and } \frac{d^2y}{dt^2} = \frac{d^2y}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{dy}{ds}\,\frac{d^2s}{dt^2},$$

$$R = m\left(\frac{dx}{ds}\,\frac{d^2y}{ds^2} - \frac{dy}{ds}\,\frac{d^2x}{ds^2}\right)\left(\frac{ds}{dt}\right)^2 + m\left(X\frac{dy}{ds} - Y\frac{dx}{ds}\right)$$

$$= m\left\{\frac{2}{\rho}\int(Xdx + Ydy) + X\frac{dy}{ds} - Y\frac{dx}{ds}\right\},$$

ρ being the radius of curvature of the curve at the point (x, y).

These results might have been arrived at more directly by using normal and tangential resolutions; for by them we have at once (art. 14)

$$\frac{d^2s}{dt^2} = X\frac{dx}{ds} + Y\frac{dy}{ds},$$

$$\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 = \frac{1}{m}R - X\frac{dy}{ds} + Y\frac{dx}{ds},$$

which manifestly give immediately the results.

The same results may be more simply stated; let the accelerating effects of the component forces in directions

along the tangent and normal be denoted by S and N respectively, and the equations may be written

$$\frac{d^2s}{dt^2} = S, \quad \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 = \frac{R}{m} + N.$$

From the first of which we have by integration,

$$\left(\frac{ds}{dt}\right)^2 = 2\int S\,ds,$$

the constant being included in the integral sign; and from the second, by substitution,

$$R = m\left(\frac{2}{\rho}\int S\,ds - N\right).$$

It is to be noticed that in all cases an accelerating effect along the normal is considered positive when it tends to the concave side of the curve, and negative when it tends to the convex side.

Since the particle exerts on the curve a force equal and opposite to that which the curve exerts on the particle, therefore the moving effect of the force with which the particle presses against the curve is $-R$ and is consequently known when R is found.

If the curve be a small tube which incloses the particle, or if the particle be a small ring and the curve be a small wire along which the ring slips, the particle will never leave the curve; for however the pressure of the particle against the curve may change its direction, there will always be the curve to withstand it. But if the curve be a material curve on one side of which the particle moves, as soon as the pressure of the particle on the curve begins to tend to that side, the particle will leave the curve, because there will then be nothing to withstand the pressure. The point at which the particle, in such a case, leaves the curve will be the point at which the accelerating effect of the force exerted by the curve on the particle passes through zero and changes its algebraical sign.

COR. 1. *To determine the motion of a particle, constrained to move along a plane curve, and acted on by no forces except the constraint.*

The equations of motion in this case become

$$\frac{d^2s}{dt^2} = 0, \quad \frac{m}{\rho}\left(\frac{ds}{dt}\right)^2 = R.$$

From the first of these equations it follows that the velocity is constant; and then from the second it appears that the force of constraint always tends towards the concave side of the curve, and that its intensity from point to point varies as the curvature. The force with which the particle presses against the curve always tends to the convex side of the curve and is proportional to the curvature.

COR. 2. *To determine the motion of a heavy particle on a smooth plane curve standing in a vertical plane.*

Let the axis of x be drawn vertically downwards and the axis of y horizontal, in the plane of the curve. The equations of motion take the form

$$\frac{d^2s}{dt^2} = g\frac{dx}{ds}, \quad \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 = \frac{R}{m} - g\frac{dy}{ds},$$

g denoting the accelerating effect of gravity. Multiplying the first of these equations by $2\frac{ds}{dt}$ and integrating, we have

$$\left(\frac{ds}{dt}\right)^2 = 2gx + C,$$

C being a constant determinable from the initial circumstances; let the particle be projected with velocity v' from the point whose vertical co-ordinate is a,

$$v'^2 = 2ga + C;$$

$$\therefore \left(\frac{ds}{dt}\right)^2 - v'^2 = 2g(x - a),$$

which gives the velocity at any point. Since in this result nothing depends on the form of the curve, it appears that the change of velocity of the particle in passing from one point to another is wholly independent of the path; since also the horizontal co-ordinate does not enter, the change of velocity depends only on the vertical distance between the points.

By substitution in the second equation of motion, we get

$$R = m\left\{\frac{v'^2 + 2g(x-a)}{\rho} + g\frac{dy}{ds}\right\},$$

which gives the force of constraint acting on the particle.

To consider some particular cases: (1) let the curve be a straight line inclined to the horizon at an angle i, the equations are

$$\frac{d^2 s}{dt^2} = g\sin i, \quad 0 = \frac{R}{m} - g\cos i,$$

R being supposed to tend towards the upper side of the path. The motion is therefore uniformly accelerated, and the constraint on the particle is constant. The moving effect of the pressure exerted by the particle on the line is $= mg\cos i$. If s be measured from the point where the motion begins from rest

$$\frac{ds}{dt} = gt\sin i, \quad \text{and } s = \frac{gt^2}{2}\sin i.$$

Hence the time of describing any length of path s from rest is $= \sqrt{\dfrac{2s}{g\sin i}}$. If the path be a chord in a circle of radius r drawn from the highest point, the length of the chord will be $2r\sin i$, and the time which the particle would take to fall from rest from the higher to the lower end of the chord would be $= \sqrt{\dfrac{4r\sin i}{g\sin i}} = \sqrt{\dfrac{4r}{g}}$, which is independent of i. If again the path were a chord of the circle terminating in the lowest point, the same result is arrived at for the time of the particle's moving along it from rest. Hence the times, required by a particle to fall from rest down all chords of a vertical circle which terminate in the highest or lowest points, are equal.

(2) Let the curve be a parabola having its axis vertical and vertex upwards. Let the vertex be origin, the parabolic axis the axis of x, and the equation of the parabola

$$y^2 = 4px,$$

$$\therefore \frac{dy}{dx} = \frac{2p}{y}, \quad \frac{d^2y}{dx^2} = -\frac{4p^2}{y^3},$$

$$\frac{dx}{ds} = \frac{y}{(4p^2 + y^2)^{\frac{1}{2}}}, \quad \frac{dy}{ds} = \frac{2p}{(4p^2 + y^2)^{\frac{1}{2}}}, \quad \frac{1}{\rho} = -\frac{4p^2}{(4p^2 + y^2)^{\frac{3}{2}}}.$$

Hence the equations of motion are

$$\frac{d^2s}{dt^2} = g\frac{dx}{ds}, \quad -\frac{4p^2}{(4p^2 + y^2)^{\frac{3}{2}}}\left(\frac{ds}{dt}\right)^2 = \frac{R}{m} - \frac{2pg}{(4p^2 + y^2)^{\frac{1}{2}}}.$$

If v' be the velocity of the particle when it is at the vertex of the parabola, the first equation gives

$$\left(\frac{ds}{dt}\right)^2 = v'^2 + 2gx = v'^2 + \frac{gy^2}{2p};$$

$$\therefore R = \frac{m}{(4p^2 + y^2)^{\frac{3}{2}}}\left\{2pg\,(4p^2 + y^2) - 4p^2\left(v'^2 + \frac{gy^2}{2p}\right)\right\}$$

$$= \frac{4p^2 m\,(2pg - v'^2)}{(4p^2 + y^2)^{\frac{3}{2}}}$$

$$= -\frac{m}{\rho}(2pg - v'^2).$$

Hence if the curve be a tube the pressure of the tube on the particle will be towards the upper or lower side of the curve according as $2pg$ is greater or less than v'^2. If $v'^2 = 2pg$ there is no force of constraint, that is, the particle moves as if it were free; this also appears from art. 24. If v'^2 be not $= 2pg$ the particle presses against the tube either continually on the side next the parabolic axis or continually on the opposite side according as v'^2 is less or greater than $2pg$ and with an intensity which from point to point varies as the curvature.

(3) Let the curve be a cycloid having its axis vertical and vertex downwards. Taking the vertex for origin and the axis of x vertically upwards, the equation of the curve is

$$y = (2rx - x^2)^{\frac{1}{2}} + r\cos^{-1}\left(\frac{r - x}{r}\right),$$

r being the radius of the generating circle. Let s be measured from the vertex; the equations of motion are

$$\frac{d^2s}{dt^2} = -g\frac{dx}{ds}, \quad \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 = \frac{R}{m} + g\frac{dy}{ds}.$$

But from the equation of the cycloid we have

$$\frac{dx}{ds} = \frac{s}{4r}, \quad \frac{dy}{ds} = \frac{(16r^2 - s^2)^{\frac{1}{2}}}{4r}, \quad \frac{1}{\rho} = -\frac{1}{(16r^2 - s^2)^{\frac{1}{2}}}.$$

The first equation of motion, on substituting, becomes

$$\frac{d^2s}{dt^2} = -\frac{gs}{4r};$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = C - \frac{gs^2}{4r},$$

C being a constant. Let the particle begin to move from rest when it is at distance a from the vertex, we have $C = \frac{ga^2}{4r}$, and therefore

$$\left(\frac{ds}{dt}\right)^2 = \frac{g}{4r}(a^2 - s^2);$$

$$\therefore \frac{ds}{(a^2 - s^2)^{\frac{1}{2}}} = -\left(\frac{g}{4r}\right)^{\frac{1}{2}} dt,$$

$$\cos^{-1}\left(\frac{s}{a}\right) = \left(\frac{g}{4r}\right)^{\frac{1}{2}} t,$$

no constant being needed since $s = a$ when $t = 0$;

$$\therefore s = a \cos\left(\frac{g}{4r}\right)^{\frac{1}{2}} t.$$

Putting $s = 0$ we get the time in which the particle falls to the vertex $= \pi\left(\frac{r}{g}\right)^{\frac{1}{2}}$, which is independent of a; therefore the particle falls to the vertex in the same time from whatever point its motion begins. The particle manifestly oscillates about the vertex, describing equal arcs on each side of it. All this might have been at once inferred from the fact that

the equation of motion is the same in form as the equation in art. 17.

Again, the second equation of motion gives, by substitution,

$$-\frac{1}{(16r^2-s^2)^{\frac{1}{2}}}\frac{g}{4r}(a^2-s^2)=\frac{R}{m}+\frac{g}{4r}(16r^2-s^2)^{\frac{1}{2}};$$

$$\therefore R=-\frac{mg}{4r(16r^2-s^2)^{\frac{1}{2}}}(a^2+16r^2-2s^2)$$

$$=\frac{mg}{4r}\left(2\rho-\frac{16r^2-a^2}{\rho}\right).$$

This quantity is always positive, therefore the action of the curve on the particle is always directed towards the concavity of the cycloid, and the particle presses on the cycloid in direction towards the convex side.

If two equal semi-cycloidal cheeks be fixed in a vertical plane with their axes vertical and connected so as to form the cusp, and if a heavy particle be hung from the cusp by a fine thread whose length is equal to the length of each of the semi-cycloidal arcs; the particle, when drawn aside from its position of rest in the plane of the cycloidal cheeks and then left to itself, will by alternate wrapping and unwrapping on the cheeks oscillate in a cycloidal arc. For the involute of the cycloidal cheeks is a cycloid, and since the particle always tends towards the convex side of its path, the string will continue stretched. A pendulum so constructed would have the property that an oscillation would be performed in the same time whatever might be the length of arc described by it in an oscillation. The resistance which the air offers to bodies moving in it will interfere with the perfect accuracy of this result. The length (l) of the pendulum would be equal to four times the length of the radius of the generating circle of the cycloid; therefore the time of an oscillation would be $=\pi\left(\frac{l}{g}\right)^{\frac{1}{2}}$.

(4) Let the curve be a circle of radius r. If the position of the particle be indicated by θ, the angle which the

radius drawn through it makes with the radius drawn vertically downwards, the equations of motion may be written

$$r\frac{d^2\theta}{dt^2} = -g\sin\theta,\quad r\left(\frac{d\theta}{dt}\right)^2 = \frac{R}{m} - g\cos\theta.$$

From the first we have by integration,

$$r\left(\frac{d\theta}{dt}\right)^2 = C + 2g\cos\theta.$$

Let v' be the velocity at the lowest point, and

$$\frac{v'^2}{r} = C + 2g;$$

$$\therefore r\left(\frac{d\theta}{dt}\right)^2 = \frac{v'^2}{r} - 2g(1-\cos\theta).$$

The time of describing any arc can be found only by the elliptic integral

$$t = \int \frac{d\theta}{\left(\frac{v'^2}{r^2} - \frac{4g}{r}\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}}.$$

From the second equation of motion

$$R = m\left(\frac{v'^2}{r} - 2g + 3g\cos\theta\right).$$

The points at which the particle comes to rest are determined by the equation

$$\frac{v'^2}{r} - 2g + 2g\cos\theta = 0;$$

there are therefore such points only when v'^2 is not greater than $4gr$. The points at which R becomes zero correspond to the values of θ which make

$$\frac{v'^2}{r} - 2g + 3g\cos\theta = 0;$$

such values exist only when v'^2 is not greater than $5gr$; it may happen however that the points given by this equation

are never reached by the particle during its motion. It is clear that the points indicated as those of no constraint will be reached by the particle only when the values of θ which correspond to such points are not greater than the values of θ, which give the points of rest, and in order to this

$$\frac{2g - \frac{v'^2}{r}}{3g} \text{ must not be less than } \frac{2g - \frac{v'^2}{r}}{2g};$$

hence $2g - \frac{v'^2}{r}$ must not be positive, that is, v'^2 must not be less than $2gr$. There are, consequently, points of no constraint situated in the path of the particle only when v'^2 is not less than $2gr$, and not greater than $5gr$; for values of v'^2 between these limits the force of constraint changes its direction at the points of no constraint. Hence if the particle be supposed to move along the inner side of a vertical ring, or to be attached to the center of the circle by a fine thread, it will not leave the curve if the velocity at the lowest point be either not greater than $(2gr)^{\frac{1}{2}}$, or not less than $(5gr)^{\frac{1}{2}}$, and the motion will be oscillatory in the former case, and continuous in the latter; if the velocity at the lowest point lie between the limits specified, the particle will leave the circle at some point of its path. If the particle be supposed to move in a tube the motion will be oscillatory if v'^2 be less than $4gr$, and continuous if greater; if $v'^2 = 4gr$ the particle will reach the highest point of the tube, and there rest.

If the particle be supposed to oscillate about the lowest point of the circle, and to describe arcs of vibration so small that their cubes and higher powers may be omitted in the equations of motion, the first equation of motion may be written

$$\frac{d^2\theta}{dt^2} + \frac{g}{r}\theta = 0.$$

This is the same in form as the equation for a cycloidal oscillation. The time of an oscillation is $= \pi \left(\frac{r}{g}\right)^{\frac{1}{2}}$; if therefore a heavy particle vibrate in a circle through infinitely

small arcs about the lowest point the oscillations are isochronous.

COR. 3. *To find the form of a plane curve standing in a vertical plane such that a heavy particle moving along it may fall to the lowest point in the same time from whatever point the motion begins.*

By a method in all respects similar to that of article 18, it appears that the weight of the particle resolved along the curve must produce an acceleration of motion towards the lowest point of the curve proportional to the length of path intercepted between the position of the particle and the lowest point. Hence if s be this length of path, we must have

$$\frac{dx}{ds} \propto s = \frac{s}{4a},$$

the axis of x being supposed to be drawn vertically upwards from the lowest point, and $4a$ being a constant;

$$\therefore\ x = \frac{s^2}{8a},$$

no constant being added, since $s = 0$ when $x = 0$; and

$$\frac{ds}{dx} = \sqrt{\frac{2a}{x}}$$

$$\text{or} \quad 1 + \left(\frac{dy}{dx}\right)^2 = \frac{2a}{x},$$

$$dy = \sqrt{\frac{2a - x}{x}}\,dx = \frac{2a - x}{\sqrt{2ax - x^2}}\,dx$$

$$= \frac{a\,dx}{\sqrt{a^2 - (a - x)^2}} + \frac{(a - x)\,dx}{\sqrt{a^2 - (a - x)^2}};$$

$$\therefore \quad y = a\cos^{-1}\frac{a - x}{a} + \sqrt{2ax - x^2},$$

the constant being zero; the curve is therefore a cycloid having its axis vertical, and vertex downwards.

COR. 4. *To determine the motion of a particle, constrained to move in a given plane curve, and acted on by a force which tends to or from a fixed center in the plane of the curve.*

If r, θ be the polar co-ordinates of the particle at the end of the time t, referred to the fixed center as pole, and P be the accelerating effect of the force in direction towards the center; the equations of motion are

$$\frac{d^2s}{dt^2} = -P\frac{dr}{ds}, \quad \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 = \frac{R}{m} + P\frac{r\,d\theta}{ds}.$$

Multiplying the first of these equations by $2\frac{ds}{dt}$ and integrating, we get

$$\left(\frac{ds}{dt}\right)^2 = v'^2 - 2\int_{r'}^{r} P\,dr,$$

v' being the velocity of the particle at the point whose radius vector is r'. This result shews that, if P be a function of r, the change of velocity of the particle in passing from one point to another depends only on the distances of the points from the center of force, and is therefore independent of the form of the curve.

The second equation gives

$$R = m\left[\frac{1}{\rho}\left(v'^2 - 2\int_{r'}^{r} P\,dr\right) - Pr\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{-\frac{1}{2}}\right].$$

The time of describing any arc is to be found by integrating the equation for the velocity.

54. *To determine the motion of a particle, constrained to move in a curve of double curvature, and acted on by given forces.*

Let the forces acting on the particle be resolved in three directions perpendicular to one another, along the tangent to the curve in the direction of motion, along the line of intersection of the normal and osculating planes in direction towards the concave side of the curve, and along the normal to the osculating plane; let the accelerating effects of the forces in these directions be denoted by S, N, and M respectively.

Let the force of constraint on the particle, which acts perpendicularly to the tangent, be resolved in the two last mentioned directions, and let the moving effects of the components be denoted by R, Q respectively.

If m be the mass of the particle, s the length of path intercepted between a fixed point and the position of the particle at the end of the time t, and ρ the radius of absolute curvature; the equations of motion are (art. 14)

$$\frac{d^2s}{dt^2} = S, \quad \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 = N + \frac{R}{m}, \quad 0 = M + \frac{Q}{m}.$$

From the first of these, we have by integration

$$\left(\frac{ds}{dt}\right)^2 = 2\int S ds,$$

and from this by another integration is to be found the time occupied by the particle in describing any part of the curve. By substitution in the second equation, we have

$$R = m\left(\frac{2}{\rho}\int S ds - N\right),$$

and from the third equation

$$Q = -mM.$$

Hence the moving effect of the force of constraint on the particle

$$= m\left\{\left(\frac{2}{\rho}\int S ds - N\right)^2 + M^2\right\}^{\frac{1}{2}},$$

and it makes with the osculating plane an angle whose co-tangent is

$$= -\frac{\frac{2}{\rho}\int S ds - N}{M}:$$

no difficulty arises about the directions indicated by the algebraical signs, if attention be paid to the directions along which the effects of the forces are resolved.

COR. 1. *To find the force which must act on a particle in order that it may describe in a given manner a given curve, along which it is constrained to move.*

If v be the velocity of the particle when it is at a point distant from a fixed point by a length of path s, the resolved part of the force which must act on the particle in direction

of the tangent has an accelerating effect equal to $v\dfrac{dv}{ds}$, and the resolved part of the force perpendicular to the tangent may be any whatever. In this it is supposed that the particle can only move along the curve; but if it move on one side of the curve and have a tendency to fly off, the component force perpendicular to the tangent must be directed from the side of the curve on which the particle moves, and must have an intensity at least sufficient to keep the particle from flying off.

COR. 2. *Two points being given, which are neither in the same vertical line nor in the same horizontal plane; to find the curve passing through the points such that a heavy particle, constrained to move along it, may fall from one to the other in the least possible time.*

Let a straight line drawn vertically downwards from the upper point be taken for axis of x, and the horizontal plane through the same point for plane of xy. Let x, y, z be the co-ordinates of the particle at the end of the time t from the commencement of motion, and s the path described by it in that time; we have

$$\frac{d^2s}{dt^2} = g\frac{dx}{ds};$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = 2gx,$$

no constant being added since the motion begins from the higher point. The time of describing any arc

$$= \frac{1}{\sqrt{2g}}\int\frac{ds}{\sqrt{x}} = \frac{1}{\sqrt{2g}}\int\sqrt{\frac{1+\left(\dfrac{dy}{dx}\right)^2+\left(\dfrac{dz}{dx}\right)^2}{x}}\,dx,$$

and the whole time of descent from the higher to the lower point

$$= \frac{1}{\sqrt{2g}}\int_0^c\sqrt{\frac{1+\left(\dfrac{dy}{dx}\right)^2+\left(\dfrac{dz}{dx}\right)^2}{x}}\,dx,$$

c being the vertical co-ordinate of the lower point. This

expression is to be a minimum; hence, by the calculus of variations, the form of the curve must be such that the co-ordinates of any point in it satisfy the relations

$$\frac{d}{dx}\left\{\frac{\frac{dy}{dx}}{\sqrt{x\left\{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2\right\}}}\right\}=0,$$

$$\frac{d}{dx}\left\{\frac{\frac{dz}{dx}}{\sqrt{x\left\{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2\right\}}}\right\}=0.$$

The quantities included by the brackets are therefore equal to constants, C and C' suppose; and we have

$$C'\frac{dy}{dx}-C\frac{dz}{dx}=0,$$

and by integration (since, when $y=0$, $z=0$),

$$C'y-Cz=0.$$

This is the equation of a vertical plane; therefore the curve sought is a plane curve. Taking the axis of y in the plane of the curve, the time of descent from the upper to the lower point is

$$=\frac{1}{\sqrt{2g}}\int_0^c\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{x}}\,dx;$$

and, as before, the co-ordinates of a point in the curve must fulfil the condition

$$\frac{\frac{dy}{dx}}{\sqrt{x\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}}=\text{a constant quantity, }\frac{1}{\sqrt{2a}}\text{ suppose;}$$

$$\therefore\ \frac{dy}{dx}=\sqrt{\frac{x}{2a-x}}=\frac{x}{\sqrt{2ax-x^2}}$$

$$= \frac{a}{\sqrt{a^2 - (a-x)^2}} - \frac{a-x}{\sqrt{a^2 - (a-x)^2}};$$

$$\text{and } y = a\cos^{-1}\left(\frac{a-x}{a}\right) - \sqrt{2ax - x^2};$$

no constant being added since $y = 0$, when $x = 0$. The constant a is to be determined from the condition that the curve passes through the lower point.

The curve is evidently a cycloid having its axis vertical, vertex downwards, and cusp at the higher point.

55. *To determine the motion of a particle, acted on by given forces, and constrained to move along a given surface.*

Let $f(x, y, z) = 0$ be the equation of the surface relative to a system of rectangular co-ordinate axes, and let x, y, z be the co-ordinates of the particle at the end of the time t measured from a fixed epoch. Let m be the mass of the particle, and X, Y, Z the component accelerating effects of the forces acting on it parallel to the co-ordinate axes.

If R denote the moving effect of the force of constraint, which acts along the normal to the surface, the equations of motion are

$$\frac{d^2x}{dt^2} = X + \frac{R}{m}\,\frac{\dfrac{df}{dx}}{\sqrt{\left(\dfrac{df}{dx}\right)^2 + \left(\dfrac{df}{dy}\right)^2 + \left(\dfrac{df}{dz}\right)^2}};$$

$$\frac{d^2y}{dt^2} = Y + \frac{R}{m}\,\frac{\dfrac{df}{dy}}{\sqrt{\left(\dfrac{df}{dx}\right)^2 + \left(\dfrac{df}{dy}\right)^2 + \left(\dfrac{df}{dz}\right)^2}};$$

$$\frac{d^2z}{dt^2} = Z + \frac{R}{m}\,\frac{\dfrac{df}{dz}}{\sqrt{\left(\dfrac{df}{dx}\right)^2 + \left(\dfrac{df}{dy}\right)^2 + \left(\dfrac{df}{dz}\right)^2}}.$$

Multiplying these equations in order by $2\dfrac{dx}{dt}$, $2\dfrac{dy}{dt}$, $2\dfrac{dz}{dt}$, add-

ing, and integrating, we have (observing that, since the particle moves in the surface, $\frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt} + \frac{df}{dz}\frac{dz}{dt} = 0$)

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 2\int(Xdx + Ydy + Zdz);$$

the constant being included in the sign of integration. If $Xdx + Ydy + Zdz$ be a perfect differential, this gives the velocity of the particle at any point of its path.

Again, multiplying the equations in order by $\frac{df}{dx}$, $\frac{df}{dy}$, $\frac{df}{dz}$, and adding, we have

$$\left\{\text{since } \frac{d^2x}{dt^2}\frac{df}{dx} + \frac{d^2y}{dt^2}\frac{df}{dy} + \frac{d^2z}{dt^2}\frac{df}{dz}\right.$$

$$= \left(\frac{ds}{dt}\right)^2\left(\frac{d^2x}{ds^2}\frac{df}{dx} + \frac{d^2y}{ds^2}\frac{df}{dy} + \frac{d^2z}{ds^2}\frac{df}{dz}\right),$$

ds being an element of the particle's path$\}$

$$\left(\frac{ds}{dt}\right)^2\left(\frac{d^2x}{ds^2}\frac{df}{dx} + \frac{d^2y}{ds^2}\frac{df}{dy} + \frac{d^2z}{ds^2}\frac{df}{dz}\right) = X\frac{df}{dx} + Y\frac{df}{dy} + Z\frac{df}{dz}$$

$$+ \frac{R}{m}\sqrt{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2 + \left(\frac{df}{dz}\right)^2}.$$

If ρ be the radius of curvature of the normal section of the surface passing through ds

$$\frac{1}{\rho} = \frac{\frac{d^2x}{ds^2}\frac{df}{dx} + \frac{d^2y}{ds^2}\frac{df}{dy} + \frac{d^2z}{ds^2}\frac{df}{dz}}{\sqrt{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2 + \left(\frac{df}{dz}\right)^2}}.$$

Hence, by substitution we get

$$R = m\left\{\frac{2}{\rho}\int(Xdx + Ydy + Zdz) - \frac{X\frac{df}{dx} + Y\frac{df}{dy} + Z\frac{df}{dz}}{\sqrt{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2 + \left(\frac{df}{dz}\right)^2}}\right\},$$

which gives the moving effect of the force of constraint in direction towards the side of the surface on which the path is concave. If the particle be moving on one side of the surface it will fly off from the surface at any point where R becomes zero and changes its algebraical sign.

In order to find the path traced out by the particle on the surface, we have from the equations of motion

$$\frac{\frac{d^2x}{dt^2} - X}{\frac{df}{dx}} = \frac{\frac{d^2y}{dt^2} - Y}{\frac{df}{dy}} = \frac{\frac{d^2z}{dt^2} - Z}{\frac{df}{dz}};$$

from which, by help of the equation

$$\left(\frac{ds}{dt}\right)^2 = 2\int(Xdx + Ydy + Zdz)$$

if necessary, t must be eliminated; the result will be a differential equation, which on integration will be the equation of a surface intersecting the given surface $f = 0$ in the curve described by the particle.

If S denote the component accelerating effect of the forces on the particle in direction along ds, and N the component accelerating effect along the normal to the surface, the results arrived at above are manifestly equivalent to

$$\left(\frac{ds}{dt}\right)^2 = 2\int S\,ds, \quad R = m\left(\frac{2}{\rho}\int S\,ds - N\right).$$

COR. 1. If $Xdx + Ydy + Zdz$, or the equivalent expression Sds, be a pefect differential, the change in the velocity of the particle in passing from one point of its path to another is wholly independent of the path, and is the same as if the particle had been constrained to move along any curve joining the points; for the change of (velocity)^2 depends on

$$2\int(Xdx + Ydy + Zdz) \text{ or } 2\int S\,ds$$

taken between the limits corresponding to the points, and is independent of the form of the intermediate curve. The forces are always supposed to be functional of the position of the particle, and in no way depending on time.

If the forces tend to or from fixed points and have intensities depending on the distances from these points, let r, $r_{,}$... be the distances of the particle from the points; and let $\phi(r)$, $\phi_{,}(r_{,})$, ... denote the accelerating effects of the forces. The quantity $Xdx + Ydy + Zdz$, or Sds, is evidently equivalent to

$$\left\{\phi(r)\frac{dr}{ds} + \phi_{,}(r_{,})\frac{dr_{,}}{ds} + \ldots\right\} ds = \phi(r)\,dr + \phi_{,}(r_{,})\,dr_{,} + \ldots;$$

and is therefore a perfect differential; this is the case of the forces in nature.

COR. 2. *To determine the motion of a particle constrained to move along a given surface and acted on by no forces.*

Since no force acts on the particle except the force of constraint, therefore the velocity of the particle is constant,

and, consequently, $\dfrac{d^2x}{dt^2} : \dfrac{d^2y}{dt^2} : \dfrac{d^2z}{dt^2} = \dfrac{d^2x}{ds^2} : \dfrac{d^2y}{ds^2} : \dfrac{d^2z}{ds^2}$.

Hence from the equations of motion we have

$$\frac{\dfrac{d^2x}{ds^2}}{\dfrac{df}{dx}} = \frac{\dfrac{d^2y}{ds^2}}{\dfrac{df}{dy}} = \frac{\dfrac{d^2z}{ds^2}}{\dfrac{df}{dz}}.$$

Now these relations express that the radius of absolute curvature at a point of the curve coincides with the normal to the surface at the same point; which is the distinctive property of the curve of maximum or minimum length drawn on the surface between two points*. Therefore if

* That this is the property of the maximum or minimum line joining two points on the surface is easily shewn; for the length of curve is

$$= \int ds = \int \sqrt{dx^2 + dy^2 + dz^2},$$

between limits corresponding to the points. The variation of this is

$$= \int \delta\sqrt{dx^2 + dy^2 + dz^2} = \int \frac{dx\,\delta dx + dy\,\delta dy + dz\,\delta dz}{\sqrt{dx^2 + dy^2 + dz^2}},$$

$$= \int \left(\frac{dx}{ds} d\delta x + \frac{dy}{ds} d\delta y + \frac{dz}{ds} d\delta z\right),$$

$$= \frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z - \int \left(\frac{d^2x}{ds^2}\delta x + \frac{d^2y}{ds^2}\delta y + \frac{d^2z}{ds^2}\delta z\right) ds.$$

any two points be taken in the curve along which the particle moves, the length of path intercepted between them is a maximum or minimum of the curves which can be drawn on the surface to join the points. If the particle can be projected from one of the points in one direction only so as to pass through the other point, the path will be the absolutely least line which can be drawn on the surface between the points; but if the particle may pass from one point to the other in more ways than one, the paths will be maxima and minima alternately, and the minima paths will be the absolutely shortest lines on the surface between the points.

COR. 3. If the surface be a surface of revolution, the equations of motion may be conveniently expressed otherwise than as in the proposition. Let the axis of the surface be the axis of z, and at the time t let r be the distance of the particle from the axis of z, and θ the angle which r makes with the plane of zx. If the forces be estimated by component accelerating effects, Z parallel to the axis of z, P along r in direction from the axis, and Q perpendicular to the plane of zr in direction of increase of θ, the equations of motion may be written, (art. 14)

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = P + \frac{R}{m}\frac{dz}{d\sigma},$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = Q,$$

$$\frac{d^2z}{dt^2} = Z - \frac{R}{m}\frac{dr}{d\sigma},$$

in which $d\sigma$ is an element of the generating curve of the surface passing through the particle.

The integrated part vanishes at each limit. Since the curve is on the surface, $\therefore \frac{df}{dx}\delta x + \frac{df}{dy}\delta y + \frac{df}{dz}\delta z = 0$; hence substituting for δz, the variation is

$$= \int\left\{\left(\frac{df}{dx}\frac{d^2z}{ds^2} - \frac{df}{dz}\frac{d^2x}{ds^2}\right)\delta x + \left(\frac{df}{dy}\frac{d^2z}{ds^2} - \frac{df}{dz}\frac{d^2y}{ds^2}\right)\delta y\right\}\frac{ds}{\frac{df}{dz}}.$$

This must be zero for the curve of maximum or minimum length, which requires that the coefficients of δx and δy under the sign of integration vanish, since δx and δy are perfectly independent. These coefficients equated to zero give the relations.

From these equations (multiplying them in order by $\frac{dr}{dt}$, $\frac{rd\theta}{dt}$, $\frac{dz}{dt}$, adding, and integrating) we shall have, as before, the (velocity)^2 or

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 2\int(Pdr + Qrd\theta + Zdz).$$

Again, multiplying the first equation by $\frac{dz}{d\sigma}$, the third by $\frac{dr}{d\sigma}$, and subtracting, we get

$$R = m\left\{\left(\frac{d\sigma}{dt}\right)^2\left(\frac{dz}{d\sigma}\frac{d^2r}{d\sigma^2} - \frac{dr}{d\sigma}\frac{d^2z}{d\sigma^2}\right) - r\frac{dz}{d\sigma}\left(\frac{d\theta}{dt}\right)^2 - P\frac{dz}{d\sigma} + Z\frac{dr}{d\sigma}\right\};$$

or, if ρ', $\rho_{,}$ be the radii of curvature in the principal sections of the surface, and v', $v_{,}$ be the component velocities of the particle parallel to these sections,

$$R = m\left(\frac{v'^2}{\rho'} - \frac{v_{,}^2}{\rho_{,}} - P\frac{dz}{d\sigma} + Z\frac{dr}{d\sigma}\right),$$

the same result as before, since the first two terms in the bracket are equivalent to (velocity)^2 ÷ the radius of curvature in the normal section of the surface which passes through the tangent to the path; the terms have different signs because the principal curvatures lie in opposite directions, $\frac{d^2r}{d\sigma^2}$ being supposed positive, and therefore the generating curve convex towards the axis of revolution.

Multiplying the third equation of motion by $\frac{dz}{dr}$ and adding it to the first, we have $\left\{\text{since } \frac{d^2z}{dt^2} = \frac{d^2z}{dr^2}\left(\frac{dr}{dt}\right)^2 + \frac{dz}{dr}\frac{d^2r}{dt^2}\right\}$,

$$\left\{1 + \left(\frac{dz}{dr}\right)^2\right\}\frac{d^2r}{dt^2} + \frac{dz}{dr}\frac{d^2z}{dr^2}\left(\frac{dr}{dt}\right)^2 - r\left(\frac{d\theta}{dt}\right)^2 = P + Z\frac{dz}{dr}.$$

Let the equation of the generating curve be $z = \phi(r)$ and

let ϕ' and ϕ'' denote $\frac{dz}{dr}$ and $\frac{d^2z}{dr^2}$ respectively. Again, from the second equation of motion,

$$r^2\frac{d\theta}{dt}\frac{d}{d\theta}\left(r^2\frac{d\theta}{dt}\right) = Qr^3;$$

$$\therefore \left(r^2\frac{d\theta}{dt}\right)^2 = 2\int Qr^3 d\theta.$$

Let H^2 denote this quantity, $\therefore H\dfrac{dH}{d\theta} = Qr^3$, and

$$\frac{dr}{dt} = -\frac{d}{d\theta}\left(\frac{1}{r}\right)r^2\frac{d\theta}{dt} = -H\frac{d}{d\theta}\left(\frac{1}{r}\right),$$

$$\frac{d^2r}{dt^2} = -\frac{H}{r^2}\left\{\frac{dH}{d\theta}\frac{d}{d\theta}\left(\frac{1}{r}\right) + H\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\right\}$$

$$= -Qr\frac{d}{d\theta}\left(\frac{1}{r}\right) - \frac{H^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right),$$

and by substitution,

$$(1+\phi'^2)\left\{Qr\frac{d}{d\theta}\left(\frac{1}{r}\right) + \frac{H^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\right\}$$

$$-\phi'\phi'' H^2\left\{\frac{d}{d\theta}\left(\frac{1}{r}\right)\right\}^2 + \frac{H^2}{r^3} + P + Z\phi' = 0,$$

which, by substitution, and reduction, becomes

$$(1+\phi'^2)\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} - r^2\phi'\phi''\left\{\frac{d}{d\theta}\left(\frac{1}{r}\right)\right\}^2$$

$$+\frac{(1+\phi'^2)\,Qr^3\dfrac{d}{d\theta}\left(\dfrac{1}{r}\right) + r^2(P + Z\phi')}{2\int Qr^3 d\theta} = 0.$$

This is the differential equation of the projection on the plane of xy of the path described by the particle; if a right cylinder have this projection for directrix, its intersection with the surface of revolution will be the path.

If the forces acting on the particle be in the plane of the

generating curve passing through the particle, $Q = 0$, and $2\int Qr^3 d\theta$ = a constant. If the only force acting on the particle be gravity, and the surface of revolution have its axis vertical, $Q = 0$, $2\int Qr^3\,d\theta$ = constant, $P = 0$, and $Z = -g$ supposing the axis of z measured upwards.

COR. 4. *A heavy particle moves on a surface of revolution, which stands with its axis vertical, in such a manner that the orbit is nearly a circle in a plane perpendicular to the axis; to find approximately the equation of the projection, on the horizontal plane, of the orbit.*

Supposing the axis of z to coincide with the axis of the surface and to be drawn upwards, and using the notation of the last cor. the equations of motion are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{R}{m}\frac{dz}{d\sigma},\quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0,\quad \frac{d^2z}{dt^2} = -g - \frac{R}{m}\frac{dr}{d\sigma}.$$

If $z = \phi(r)$ be the equation of the generating curve, we have, multiplying the third equation by $\frac{dz}{dr}$ or $\phi'(r)$ and adding it to the first,

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d^2z}{dt^2} + g\right)\phi'(r) = 0.$$

From the second equation $r^2\frac{d\theta}{dt}$ = a constant, h suppose. Now let r be $= a$ at an apse and $= a + x$ at the time t, and let the velocity at the apse be $= (1 + n) \times$ the velocity in a circle at the same distance; suppose x and n to be quantities so small that higher powers of them than the first may be neglected. Hence

$$\frac{d\theta}{dt} = \frac{h}{(a+x)^2} = \frac{h}{a^2}\left(1 - 2\frac{x}{a}\right),$$

$$\frac{d^2r}{dt^2} = \frac{d^2x}{dt^2} = \frac{d}{d\theta}\left(\frac{dx}{d\theta}\frac{d\theta}{dt}\right)\frac{d\theta}{dt} = \frac{h^2}{a^4}\frac{d^2x}{d\theta^2},$$

$$z = \phi(a + x) = \phi(a) + \phi'(a)\,x,$$

$$\frac{d^2z}{dt^2} = \frac{h^2}{a^4}\phi'(a)\frac{d^2x}{d\theta^2},$$

$$\phi'(r) = \phi'(a+x) = \phi'(a) + \phi''(a)\,x,$$

$$\text{and } r\left(\frac{d\theta}{dt}\right)^2 = \frac{h^2}{(a+x)^3} = \frac{h^2}{a^3}\left(1 - 3\frac{x}{a}\right).$$

Substituting these in the above equation, it becomes

$$\frac{h^2}{a^4}\{1 + \overline{\phi'(a)}|^2\}\frac{d^2x}{d\theta^2} + \left\{3\frac{h^2}{a^4} + g\phi''(a)\right\}x = \frac{h^2}{a^3} - g\,\phi'(a).$$

But if the particle described a circle of radius a in a horizontal plane, we should have

$$g + \frac{R}{m}\frac{dr}{d\sigma} = 0, \quad \therefore \quad \frac{R}{m}\frac{dz}{d\sigma} = -g\frac{dz}{dr} = -g\phi'(a),$$

or the accelerating effect of the central force would $= g\phi'(a)$. Hence $h^2 = a^2 \times (\text{velocity})^2$ at apse,

$$= a^2(1+n)^2 \times (\text{velocity})^2 \text{ in circle at distance } a,$$
$$= a^2(1+2n)\,ag\phi'(a), \quad (\text{art. } 29)$$
$$= ga^3\phi'(a)(1+2n),$$

substituting this value of h^2, and omitting products of x and n we get

$$\{1 + \overline{\phi'(a)}|^2\}\frac{d^2x}{d\theta^2} + \left\{3 + \frac{a\phi''(a)}{\phi'(a)}\right\}x = 2na.$$

The solution of this equation is easily found to be

$$x = \frac{2na}{3 + \dfrac{a\phi''(a)}{\phi'(a)}}\left\{1 - \cos\left\{\theta\sqrt{\frac{3 + \dfrac{a\phi''(a)}{\phi'(a)}}{1 + \overline{\phi'(a)}|^2}}\right\}\right\}$$

the constants being determined so that x and $\dfrac{dx}{d\theta}$ may each be $= 0$ when $\theta = 0$. The approximate equation of the projection of the path is therefore

$$r = a\left\{1 + \frac{4n}{3 + \dfrac{a\phi''(a)}{\phi(a)}}\sin^2\left\{\frac{\theta}{2}\sqrt{\frac{3 + \dfrac{a\phi''(a)}{\phi'(a)}}{1 + \overline{\phi'(a)}|^2}}\right\}\right\}$$

The apsidal distances in this orbit are a and $a\left\{1+\dfrac{4n}{3+\dfrac{a\phi''(a)}{\phi'(a)}}\right\}$,

and the apsidal angle is evidently

$$= \pi\left\{\frac{1+\overline{\phi'(a)}|^2}{3+\dfrac{a\phi''(a)}{\phi'(a)}}\right\}^{\frac{1}{2}}.$$

If, for example, the surface be a sphere of radius c, we have

$$\phi(a) = c - \sqrt{c^2 - a^2},$$

$$\phi'(a) = \frac{a}{\sqrt{c^2 - a^2}};$$

and taking the logarithmic differential of each side

$$\frac{\phi''(a)}{\phi'(a)} = \frac{1}{a} + \frac{a}{c^2 - a^2},$$

$$\frac{a\phi''(a)}{\phi'(a)} = \frac{c^2}{c^2 - a^2};$$

$$\therefore \text{ the apsidal angle } = \pi\sqrt{\frac{1+\dfrac{a^2}{c^2-a^2}}{3+\dfrac{c^2}{c^2-a^2}}} = \frac{\pi c}{\sqrt{4c^2 - 3a^2}}.$$

56. *If a particle acted on by forces be constrained to move along a surface, the action which it accumulates in passing from one point of its path to another is generally less than if it had been constrained to move along any other curve drawn on the surface between the points.*

If v be the velocity of a particle, $v\,ds$ is called its *action* in passing over an element of path ds, and the accumulated action in passing from one point to another of the path is $\int v\,ds$, taken between limits corresponding to the points. It is to be proved that, of all curves, drawn on the surface between the points, along which the particle may be constrained to move under the action of the forces, $\int v\,ds$ is a

maximum or minimum for the curve along which the particle actually does move; in other words, that $\delta\int v\,ds = 0$ for this curve.

$$\text{Now } \delta\int v\,ds = \int\delta(v\,ds) = \int(\delta v\,ds + v\,\delta ds),$$
$$= \int(\delta v\,ds + v\,d\delta s),$$
$$= v\,\delta s \text{ (between the limits)} + \int(\delta v\,ds - dv\,\delta s).$$

But $\delta s = 0$ at each limit, since the curve and varied curve have the same extreme points,

$$\therefore\ \delta\int v\,ds = \int(\delta v\,ds - dv\,\delta s,$$
$$= \int\left(v\,\delta v - \frac{dv}{dt}\,\delta s\right)dt.$$

If S be the component accelerating effect along ds of the forces acting on the particle

$$\frac{dv}{dt} = S,$$

and if $S\,ds$ be a perfect differential (dK suppose),

$$v^2 = 2\int S\,ds = 2(C + K),$$

in which C is a constant, and K functional of the position of the particle. But this equation holds good for the varied path,

$$\therefore\ v\,\delta v = \delta K = \frac{dK}{ds}\,\delta s,$$
$$= S\,\delta s.$$

$$\text{Hence}\quad v\,\delta v - \frac{dv}{dt}\,\delta s = 0,$$

$$\text{and}\quad \delta\int v\,ds = 0,$$

which proves the proposition.

If the particle can be projected, with a given velocity from one of the points, in one direction only, so as to reach the other point, $\int v\,ds$ is evidently a minimum; but if there are more directions than one along which, if the particle be projected, it will pass through the other point, $\int v\,ds$ will be

alternately a maximum and minimum for the different paths taken in order.

It may be shewn in a similar way, that if a particle move freely under the action of forces functional of its position, the accumulated action in passing from one point to another of its path will be less than if it had been constrained to move between the points along any other path,—provided there is only one direction in which the particle can be projected from the first point, with the velocity which it has there, so that by moving freely under the action of the forces it may reach the second point. If there be more than one such direction, the accumulated action will be alternately a maximum and minimum for the paths corresponding to the different directions taken in order. The only difference between the proof for a particle moving freely and that for a particle constrained to move along a surface, is that in the former case the varied paths are perfectly arbitrary while in the latter they are subjected to be drawn on the surface.

If a particle move along a surface under the action of no forces, the velocity is constant and therefore $\int ds$ or the length of path is a maximum or minimum; which agrees with the result of cor. 2 of last article.

57. If the curve along which a particle is constrained to move be rough, a force of friction will operate on the particle in a direction contrary to the direction of motion. If S, N, and M be the component accelerating effects of the forces acting on the particle in the respective directions along the tangent in direction of the motion, along the radius of absolute curvature, and along the perpendicular to the osculating plane; and if $-F$, R, and Q be the component moving effects, resolved in the same directions respectively, of the force of constraint on the particle, the equations of motion are

$$\frac{d^2s}{dt^2} = S - \frac{F}{m}, \quad \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 = N + \frac{R}{m}, \quad 0 = M + \frac{Q}{m},$$

m being the mass of the particle, s the length of path intercepted between a fixed point and the position of the particle

at the end of the time t, and ρ the radius of absolute curvature at the extremity of s. It is usual to suppose that

$$F = \mu \sqrt{R^2 + Q^2},$$

μ being a quantity depending only on the materials of which the particle and curve are composed, and wholly independent of the velocity. From these equations and the equations of the curve everything connected with the motion is to be determined.

If a particle move on a rough surface, let $f(xyz) = 0$ be the equation of the surface, X, Y, Z the component accelerating effects of the forces acting on the particle in directions of the co-ordinate axes, and let R and $-F$ be the component moving effects of the force of constraint in directions along the normal to the surface and the tangent to the path of the particle. The equations of motion are

$$\frac{d^2x}{dt^2} = X - \frac{F}{m}\frac{dx}{ds} + \frac{R}{m}\frac{\frac{df}{dx}}{\sqrt{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2 + \left(\frac{df}{dz}\right)^2}},$$

$$\frac{d^2y}{dt^2} = Y - \frac{F}{m}\frac{dy}{ds} + \frac{R}{m}\frac{\frac{df}{dy}}{\sqrt{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2 + \left(\frac{df}{dz}\right)^2}},$$

$$\frac{d^2z}{dt^2} = Z - \frac{F}{m}\frac{dz}{ds} + \frac{R}{m}\frac{\frac{df}{dz}}{\sqrt{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2 + \left(\frac{df}{dz}\right)^2}};$$

x, y, z being the co-ordinates of the particle at the time t, ds an element of path, and m the particle's mass. It is usual to suppose here also that

$$F = \mu R,$$

μ depending only on the nature of the particle and the surface. It is probable that the effect of a component constraint

along the surface perpendicular to the path ought to have been considered as existing; for the force of friction acts in direction contrary to the direction of a tendency to motion, and the direction of tendency to motion may differ from that of the actual motion on a rough surface; this is rendered likely from the known fact that the friction which opposes a beginning of motion is greater (for a given normal pressure) than that which opposes an existing motion.

So little is known concerning the force of friction which acts on particles moving on rough curves and surfaces, that the foregoing equations are not to be depended upon; it would therefore be needless to gather results from them.

58. *To determine the motion of a particle, acted on by given forces, and constrained to move along a smooth plane curve which revolves uniformly about an axis in its plane.*

Let the axis about which the curve revolves be taken for axis of z; and at the end of the time t let z be the distance of the particle from the plane of xy, r its distance from the axis of z, and θ the angle which r makes with the plane of zx. Since the curve revolves uniformly, the velocity of increase of θ is constant, let therefore $\frac{d\theta}{dt} = \omega$ a constant. Let m be the mass of the particle, and let the component accelerating effects of the forces acting on it be Z along z, P along r, and Q perpendicular to the plane of zr.

If the force of constraint on the particle, which acts perpendicularly to the revolving curve, have component moving effects R in the plane of the curve and R' perpendicular to it, the equations of motion are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = P + \frac{R}{m}\frac{dz}{d\sigma}, \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = Q + \frac{R'}{m},$$

$$\frac{d^2z}{dt^2} = Z - \frac{R}{m}\frac{dr}{d\sigma},$$

in which $d\sigma$ is an element of the revolving curve.

From the first and third equations, we have

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - P + \left(\frac{d^2z}{dt^2} - Z\right)\frac{dz}{dr} = 0.$$

$$\text{Now } \frac{d^2r}{dt^2} = \frac{d\theta}{dt}\frac{d}{d\theta}\left(\frac{dr}{d\theta}\frac{d\theta}{dt}\right) = \omega^2\frac{d^2r}{d\theta^2}.$$

If $z = \phi(r)$ be the equation of the revolving curve, $\dfrac{dz}{dr} = \phi'(r)$, and $\dfrac{d^2z}{dr^2} = \phi''(r)$, we have

$$\frac{d^2z}{dt^2} = \frac{d\theta}{dt}\frac{d}{d\theta}\left\{\omega\phi'(r)\frac{dr}{d\theta}\right\} = \omega^2\left\{\phi''(r)\left(\frac{dr}{d\theta}\right)^2 + \phi'(r)\frac{d^2r}{d\theta^2}\right\}.$$

Hence by substitution

$$\frac{d^2r}{d\theta^2}\{1 + \overline{\phi'(r)}|^2\} + \phi'(r)\phi''(r)\left(\frac{dr}{d\theta}\right)^2 - r - \frac{P + Z\phi'(r)}{\omega^2} = 0.$$

This is the differential equation of the curve which is the projection of the particle's path on the plane of xy; when integrated it will give r in terms of θ, and then z becomes known, also $\theta = \omega t +$ constant. Thus the position of the particle is known at any instant.

For example, let the curve be a straight line making angle a with the axis, which is vertical, and let gravity be the only force acting. Considering the axis measured upwards from the point of intersection with the revolving line, we have

$$z = r\cot a;$$

$$\therefore\ \phi'(r) = \cot a, \quad \phi''(r) = 0; \quad \text{also } P = 0,\ Z = -g.$$

Hence the equation above becomes

$$\frac{d^2r}{d\theta^2} - r\sin^2 a + \frac{g}{\omega^2}\sin a\cos a = 0,$$

the integral of which is

$$r - \frac{g}{\omega^2}\cot a = C\epsilon^{\theta\sin a} + C'\epsilon^{-\theta\sin a},$$

C and C' being constants; to determine them, suppose that

when $t = 0$, $\theta = 0$, and the particle is at the intersection of the line with the axis, and moving along the line with velocity v;

$$\therefore -\frac{g}{\omega^2}\cot\alpha = C + C',$$

$$\frac{v\,\mathrm{cosec}\,\alpha}{\omega} = C - C'.$$

The equation of the projection on the plane of xy of the path is therefore

$$r = \frac{2g\cos\alpha + (v\omega - g\cos\alpha)\,\epsilon^{\theta\sin\alpha} - (v\omega + g\cos\alpha)\,\epsilon^{-\theta\sin\alpha}}{2\omega^2\sin\alpha}.$$

Also $\theta = \omega t$ and $z = r\cot\alpha$; the position of the particle at the time t is thus determined.

59. *To determine the motion of two particles, constrained to be always at the same distance from one another, and acted on by no forces except the constraints.*

The particles may be supposed connected together by a rigid rod without mass. The forces of constraint on the particles act along the rod in opposite directions and with equal intensities, by the third law of motion.

Since the particles are always at the same distance from one another, it is clear that their component relative velocity in direction of the line joining them must be zero. If the particles be projected with velocities which do not satisfy this condition, an impulsive force of constraint must act in order to change these velocities to others which satisfy the condition.

Let m, m' be the masses of the particles, and let them be projected with velocities u, u' in directions making angles β, β' with the line passing through their initial positions. Suppose that $u\cos\beta - u'\cos\beta'$ is not equal to zero. An impulsive constraint is immediately called into action; let R denote the impulsive effect of this constraint on each of the particles, and after the action of this force let the particles move off with velocities v, v' along lines making angles α, α'

with the line passing through the initial positions of the particles, these velocities being such that

$$v \cos \alpha - v' \cos \alpha' = 0.$$

Hence we have

$$mv \cos \alpha = mu \cos \beta - R, \quad m'v' \cos \alpha' = m'u' \cos \beta' + R,$$
$$v \sin \alpha = u \sin \beta, \qquad v' \sin \alpha' = u' \sin \beta'.$$

From these five equations R, v, v', α, α' are known. If j and i be the inclination to one another of the lines of motion before and after the impulse, we have (since the planes passing through the line joining the initial positions of the particles and the respective lines of motion continue unchanged, and the cosine of the angle between these planes is equal to)

$$\frac{\cos j - \cos \beta \cos \beta'}{\sin \beta \sin \beta'} = \frac{\cos i - \cos \alpha \cos \alpha'}{\sin \alpha \sin \alpha'},$$

which gives i.

The relative plane of motion of the particles is to be found at any instant as in article 44. The particles m, m' will manifestly describe circles in the relative plane of motion about their center of gravity as common center, with the respective constant velocities, (art. 44, cor. 2)

$$\frac{m'(v^2 + v'^2 - 2vv' \cos i)^{\frac{1}{2}}}{m + m'}, \quad \frac{m(v^2 + v'^2 - 2vv' \cos i)^{\frac{1}{2}}}{m + m'}.$$

The force of constraint on each particle acts in direction towards the center of gravity, with moving effect (art. 29)

$$= \frac{mm'(v^2 + v'^2 - 2vv' \cos i)}{r(m + m')},$$

in which r is the distance between the particles.

Since this expression is always positive, therefore the force of constraint on each particle is always directed towards the center of gravity, and the force which each particle exerts on the constraining rod is always directed from the center of gravity. The requisite constraint of the particles will consequently be effected by connecting them together

by an inextensible thread instead of by a rigid rod; for if the thread be stretched initially, it will continue stretched throughout the motion.

60. *Two straight tubes, fixed with their upper ends together and at given inclinations to the horizon, contain heavy particles, connected by an inextensible string which passes along the tubes and over their point of junction; to determine the motion of the particles.*

Let i, i' be the angles of inclination of the tubes to the horizon, m, m' the masses of the particles, and c the length of the string which joins them.

At the time t, let the particle m be at distance s from the point of junction of the tubes, let T be the moving effect of the force which the string exerts on each of the particles, and let R, R' be the moving effects of the forces of constraint exerted by the tubes on the particles.

The equations of motion of m are

$$m\frac{d^2s}{dt^2} = mg\sin i - T, \quad 0 = mg\cos i - R,$$

and those of m' are

$$m'\frac{d^2(c-s)}{dt^2} = m'g\sin i' - T, \quad 0 = m'g\cos i' - R'.$$

From which we have

$$\frac{d^2s}{dt^2} = \frac{m\sin i - m'\sin i'}{m+m'}g.$$

The motion is therefore uniformly accelerated. The tension of the string has a moving effect

$$T = \frac{mm'(\sin i + \sin i')}{m+m'}g.$$

61. *Two heavy particles are hung from a fixed point by means of a fine thread at different points of which they are fastened; to determine the motion of the particles when they are slightly disturbed in a vertical plane passing through the point of suspension.*

Let m be the mass of the upper and m' that of the lower particle, a the length of thread between m and the point of suspension, and a' the length of thread between m and m'.

At the end of the time t measured from a fixed epoch let the threads a, a' make angles θ, θ' with the vertical, and let them be stretched by forces whose moving effects are T, T' respectively. Estimating accelerating effects vertically and horizontally in the plane of motion, we have for m,

$$m\frac{d^2}{dt^2}(a\cos\theta) = mg - T\cos\theta + T'\cos\theta',$$

$$m\frac{d^2}{dt^2}(a\sin\theta) = - T\sin\theta + T'\sin\theta',$$

and for m',

$$m'\frac{d^2}{dt^2}(a\cos\theta + a'\cos\theta') = m'g - T'\cos\theta',$$

$$m'\frac{d^2}{dt^2}(a\sin\theta + a'\sin\theta') = - T'\sin\theta'.$$

From these four equations the quantities θ, θ', T and T' are to be found in terms of t.

The motion of the system will be vibratory, since the initial displacements of the particles from their points of rest are small. If throughout the motion θ and θ' be so small that higher powers of them than the first may be omitted in the equations of motion, we have

$$0 = mg - T + T', \quad ma\frac{d^2\theta}{dt^2} = - T\theta + T'\theta',$$

$$0 = m'g - T', \qquad m'a\frac{d^2\theta}{dt^2} + m'a'\frac{d^2\theta'}{dt^2} = - T'\theta';$$

from which we get $T' = m'g$, $T = (m + m')g$, and

$$\left\{ma\frac{d^2}{dt^2} + (m + m')g\right\}\theta - m'g\theta' = 0,$$

$$a\frac{d^2\theta}{dt^2} + \left(a'\frac{d^2}{dt^2} + g\right)\theta' = 0.$$

Eliminating θ' from these equations, we have

$$maa'\frac{d^4\theta}{dt^4} + (m+m')(a+a')g\frac{d^2\theta}{dt^2} + (m+m')g^2\theta = 0,$$

of which the complete solution is

$$\theta = c_1\cos(\sqrt{k_1}t + C_1) + c_2\cos(\sqrt{k_2}t + C_2),$$

c_1, c_2, C_1, C_2 being constants depending on the initial circumstances of the motion, and k_1, k_2 are the roots of the quadratic equation

$$maa'x^2 - (m+m')(a+a')gx + (m+m')g^2 = 0.$$

If λ_1 be put instead of

$$\frac{a}{\frac{g}{k_1} - a'} = \frac{m + m' - ma\frac{k_1}{g}}{m'},$$

and λ_2 instead of the same function of k_2, we have

$$\theta' = \lambda_1 c_1 \cos(\sqrt{k_1}\,t + C_1) + \lambda_2 c_2 \cos(\sqrt{k_2}\,t + C_2).$$

It is evident that the roots of the quadratic are both positive, so that $\sqrt{k_1}$ and $\sqrt{k_2}$ are both real quantities.

To determine the constants, let the particles be initially at rest in positions such that the strings make angles α, α' with the vertical; we have therefore

$$\alpha = c_1\cos C_1 + c_2\cos C_2,$$
$$\alpha' = \lambda_1 c_1\cos C_1 + \lambda_2 c_2\cos C_2,$$
$$0 = c_1\sqrt{k_1}\sin C_1 + c_2\sqrt{k_2}\sin C_2,$$
$$0 = \lambda_1 c_1\sqrt{k_1}\sin C_1 + \lambda_2 c_2\sqrt{k_2}\sin C_2,$$

from which we find, $\sin C_1 = 0$, $\sin C_2 = 0$,

$$c_1 = \frac{\alpha' - \lambda_2\alpha}{\lambda_1 - \lambda_2},\quad c_2 = \frac{\lambda_1\alpha - \alpha'}{\lambda_1 - \lambda_2};$$

$$\therefore \theta = \frac{1}{\lambda_1 - \lambda_2}\left\{(\alpha' - \lambda_2\alpha)\cos\sqrt{k_1}t + (\lambda_1\alpha - \alpha')\cos\sqrt{k_2}t\right\},$$

$$\theta' = \frac{1}{\lambda_1 - \lambda_2}\left\{\lambda_1(\alpha' - \lambda_2\alpha)\cos\sqrt{k_1}\,t + \lambda_2(\lambda_1\alpha - \alpha')\cos\sqrt{k_2}t\right\}.$$

CHAPTER VI.

THE MOTION OF A PARTICLE IN A RESISTING MEDIUM.

---

63. If a body move through a material medium, such as air, water, or any yielding substance, a resistance is offered to the motion; for the body, in displacing the particles of the medium situated in or about its path, exerts forces on them, and therefore (by the third law of motion) they in their turn exert on the body equal and opposite forces which tend to retard its motion. If we suppose the body to cut out for itself a tubular bore in the substance of the medium by impinging successively on the particles which lie in its path, the momentum lost by it in a given time will depend on two circumstances, viz. (1) on the number of particles impinged on, and (2) on the momentum communicated to each particle: now the number of particles impinged on in a given time will be proportional to the length of path described in the time, that is, to the velocity of the body; and the momentum communicated to each particle will also be proportional to the velocity (art. 48); therefore the momentum lost by the body in a given time, and consequently the force of resistance offered by the medium to its motion, will be proportional to the square of the velocity. The medium has been supposed of uniform density throughout, that is, it has been supposed that equal volumes of the medium contain equal masses: if the density be not uniform, the resistance will vary from point to point with the density. The result thus rudely arrived at is found to be very near the truth in the case of a body moving with a small velocity in a medium of small density, but is found to be altogether wide of the truth for bodies moving with large velocities. The true law of resistance is unknown.

As any assumed law will serve the purpose of illustration

as well as another, in the following examples of motion in a resisting medium the resistance will be supposed to vary as the square of the velocity, and the medium will be supposed of uniform density unless the contrary be stated. Although the bodies whose motions are investigated are supposed to be particles, yet the investigations will apply to the motions of the centers of gravity of bodies, such as spherical balls, provided the resultant action of the resistance is in direction opposite to the direction of the motion (prop. 1 of art. 47).

64. *To determine the motion in a resisting medium of a particle, acted on by a force in the line of motion.*

Since the force acts in the line of motion and the resistance acts in direction opposite to the direction of motion, therefore the path of the particle is a straight line. After an interval of time t elapsed from a fixed epoch let s be the distance of the particle from a fixed point in its path, and let S denote the accelerating effect of the force acting on it. If kv^2 be the accelerating effect of the force of resistance which the medium offers to the motion, k being a constant quantity and v the particle's velocity, we have for the equation of the motion

$$\frac{d^2 s}{dt^2} = S \mp k \left(\frac{ds}{dt}\right)^2,$$

the upper or the lower sign being taken according as $\frac{ds}{dt}$ is positive or negative.

Transposing and multiplying by $\epsilon^{\pm ks}$, for the purpose of integrating, we get

$$\frac{d}{dt}\left(\epsilon^{\pm ks}\frac{ds}{dt}\right) = \epsilon^{\pm ks} S,$$

and multiplying each side by $2\epsilon^{\pm ks}\frac{ds}{dt}$, and integrating

$$\left(\frac{ds}{dt}\right)^2 = 2\epsilon^{\mp 2ks}\int \epsilon^{\pm 2ks} S ds,$$

the proper constant being supposed included in the sign of

integration. The time of describing any length of path is to be found by another integration.

COR. 1. If the particle be initially projected with a velocity v', and be acted on by no force except the resistance offered by the medium, we have

$$\frac{d^2s}{dt^2} = -k\left(\frac{ds}{dt}\right)^2,$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = C\epsilon^{-2ks},$$

C being a constant. Let s be measured from the point of initial projection: when $s = 0$, $\frac{ds}{dt} = v'$, $\therefore$ $C = v'^2$, and

$$\frac{ds}{dt} = v'\epsilon^{-ks};$$

$$\therefore \epsilon^{ks}\,ds = v'dt, \quad \text{and } t = \frac{1}{kv'}(\epsilon^{ks} - 1),$$

the constant being determined by making s and t begin together. Hence

$$s = \frac{1}{k}\log(1 + kv't),$$

and the velocity at the time t, or $\frac{ds}{dt} = \frac{v'}{1 + kv't}$.

COR. 2. If the motion to be determined be that of a heavy particle projected vertically upwards with a velocity v', the equation of the upward motion is

$$\frac{d^2s}{dt^2} = -g - k\left(\frac{ds}{dt}\right)^2,$$

in which g is the accelerating effect of the weight of the particle in the medium. If the medium itself have weight, g will be less than the accelerating effect of the particle's weight in a vacuum, because the weight of a body in a heavy medium is less than its weight in a vacuum.

Hence we have

$$\left(\frac{ds}{dt}\right)^2 = -2g\epsilon^{-2ks}\int \epsilon^{2ks}\,ds = \frac{g}{k}\left(C\epsilon^{-2ks} - 1\right),$$

C being a constant; to find it, $\frac{ds}{dt} = v'$ when $s = 0$,

$$\therefore\ C = 1 + \frac{kv'^2}{g},$$

and
$$\frac{\epsilon^{ks}\,ds}{\sqrt{1 + \frac{kv'^2}{g} - \epsilon^{2ks}}} = \sqrt{\frac{g}{k}}\,dt.$$

Integrating and correcting so that s and t may begin together,

$$\sin^{-1}\frac{\epsilon^{ks}}{\sqrt{1 + \frac{kv'^2}{g}}} = \sqrt{kg}\,t + \sin^{-1}\frac{1}{\sqrt{1 + \frac{kv'^2}{g}}};$$

$$\therefore\ s = \frac{1}{k}\log\left(\cos\sqrt{kg}\,t + v'\sqrt{\frac{k}{g}}\sin\sqrt{kg}\,t\right),$$

and the velocity $= \sqrt{\frac{g}{k}}\,\dfrac{v'\sqrt{\frac{k}{g}} - \tan\sqrt{kg}\,t}{1 + v'\sqrt{\frac{k}{g}}\tan\sqrt{kg}\,t}.$

Hence the particle comes to rest at time $\frac{1}{\sqrt{kg}}\tan^{-1}v'\sqrt{\frac{k}{g}}$ after the instant of projection, and at a height $\frac{1}{2k}\log\left(1 + \frac{kv'^2}{g}\right)$ above the point of projection. The particle will then begin to descend, and the equation of its downward motion will be

$$\frac{d^2s}{dt^2} = -g + k\left(\frac{ds}{dt}\right)^2;$$

$$\therefore\quad \left(\frac{ds}{dt}\right)^2 = -2g\epsilon^{2ks}\int \epsilon^{-2ks}\,ds = \frac{g}{k}\left(C\epsilon^{2ks} + 1\right).$$

Now $\dfrac{ds}{dt}=0$ when $\epsilon^{2ks}=1+\dfrac{kv'^2}{g}$; $\therefore\ C=-\dfrac{1}{1+\dfrac{kv'^2}{g}}$, and

$$-\frac{\epsilon^{-ks}ds}{\sqrt{\epsilon^{-2ks}-\left(1+\dfrac{kv'^2}{g}\right)^{-1}}}=\sqrt{\frac{g}{k}}\,dt.$$

Integrating and supposing that t is measured from the beginning of the downward motion,

$$\log\left\{\epsilon^{-ks}\sqrt{1+\frac{kv'^2}{g}}+\sqrt{\epsilon^{-2ks}\left(1+\frac{kv'^2}{g}\right)-1}\right\}=\sqrt{kg}\,t;$$

$$\therefore\quad s=\frac{1}{k}\log\left(\frac{2\sqrt{1+\dfrac{kv'^2}{g}}}{\epsilon^{\sqrt{kg}t}+\epsilon^{-\sqrt{kg}t}}\right);$$

and the velocity downwards $\left(=-\dfrac{ds}{dt}\right)$

$$=\sqrt{\frac{g}{k}}\,\frac{\epsilon^{\sqrt{kg}t}-\epsilon^{-\sqrt{kg}t}}{\epsilon^{\sqrt{kg}t}+\epsilon^{-\sqrt{kg}t}}.$$

Hence after a time $\dfrac{1}{\sqrt{kg}}\log\left(\sqrt{1+\dfrac{kv'^2}{g}}+v'\sqrt{\dfrac{k}{g}}\right)$ from the beginning of the downward motion the particle is at the point of projection, and is moving downwards with a velocity $=\dfrac{v'}{\sqrt{1+\dfrac{kv'^2}{g}}}$. It then continues to move downwards with a continually increasing velocity; but although the velocity continually increases, it never exceeds $\sqrt{\dfrac{g}{k}}$.

COR. 3. Suppose the particle to be attracted to a fixed center by a force whose intensity is proportional to the distance between the particle and the center. If the center of force be taken for origin of distances, the accelerating effect of the force on the particle may be represented by μs in direction towards the origin. Hence the equation of motion from the positive to the negative side of the origin is

$$\frac{d^2s}{dt^2} = -\mu s + k\left(\frac{ds}{dt}\right)^2,$$

and the equation of motion from the negative to the positive side is

$$\frac{d^2s}{dt^2} = -\mu s - k\left(\frac{ds}{dt}\right)^2.$$

From the first of these equations we have

$$\left(\frac{ds}{dt}\right)^2 = -2\mu\epsilon^{2ks}\int s\epsilon^{-2ks}ds = \frac{\mu s}{k} + \frac{\mu}{2k^2} + C\epsilon^{2ks},$$

C being a constant. Let the particle be initially at rest at distance a from the origin, therefore

$$0 = \frac{\mu a}{k} + \frac{\mu}{2k^2} + C\epsilon^{2ka};$$

$$\left(\frac{ds}{dt}\right)^2 = \frac{\mu}{k}\left\{s + \frac{1}{2k} - \left(a + \frac{1}{2k}\right)\epsilon^{-2k(a-s)}\right\}.$$

The particle will come to rest at some point on the negative side of the origin; if a' be the distance of this point from the origin, $\frac{ds}{dt}$ will $= 0$ when $s = -a'$. After this the motion will be in direction from the negative to the positive side of the origin, and the second equation of motion will then apply. From it we get

$$\left(\frac{ds}{dt}\right)^2 = \frac{\mu}{k}\left\{\frac{1}{2k} - s - \left(\frac{1}{2k} + a'\right)\epsilon^{-2k(a'+s)}\right\},$$

determining the constant by making $\frac{ds}{dt} = 0$ when $s = -a'$.

This expression will be rendered again zero for some value of $s = a''$, that is, the particle will come to rest at some point on the positive side of the origin. The motion afterwards will be from positive to negative, and may be determined as before. The particle evidently oscillates about the origin, and the extreme lengths of vibration are continually diminishing.

65. *To determine the motion in a resisting medium of a particle acted on by any forces.*

Let x, y, z be the co-ordinates of the particle relative to an assumed system of rectangular axes, at the end of the time t, and let X, Y, Z be the component accelerating effects, parallel to the respective co-ordinate axes, of the forces acting on the particle. If the accelerating effect of the resistance which the medium offers to the motion be represented by $k\left(\frac{ds}{dt}\right)^2$ as before, and if $\frac{ds}{dt}$ and the component velocities of the particle be supposed positive, we have

$$\left\{\because \left(\frac{ds}{dt}\right)^2\frac{dx}{ds} = \frac{ds}{dt}\frac{dx}{dt} \text{ \&c.,}\right\}$$

$$\frac{d^2x}{dt^2} = X - k\frac{ds}{dt}\frac{dx}{dt}, \quad \frac{d^2y}{dt^2} = Y - k\frac{ds}{dt}\frac{dy}{dt},$$

$$\frac{d^2z}{dt^2} = Z - k\frac{ds}{dt}\frac{dz}{dt}.$$

From these three equations, two equations are to be found which do not involve t; when these two equations are integrated and corrected with the proper constants, they will represent two surfaces, the common section of which will be the particle's path.

Multiplying the equations in order by $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, adding, multiplying the sum by $2\epsilon^{2k\int\sqrt{dx^2+dy^2+dz^2}}$, transposing, and integrating, we have

$$\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}\epsilon^{2k\int\sqrt{dx^2+dy^2+dz^2}}$$

$$= 2\int\epsilon^{2k\int\sqrt{dx^2+dy^2+dz^2}}(X\,dx + Y\,dy + Z\,dz);$$

which, when the integration can be effected, may be of use in the above elimination of t.

COR. 1. To find the force which must act on a particle so that it may describe in a resisting medium a given path in a given manner; if v be the velocity of the particle at point (x, y, z), we have

$$\left\{\text{since } \frac{d^2x}{dt} = v\frac{d}{ds}\left(v\frac{dx}{ds}\right); \text{ and } \left(\frac{ds}{dt}\right)^2\frac{dx}{ds} = v^2\frac{dx}{ds}\right\},$$

$$X = v\frac{d}{ds}\left(v\frac{dx}{ds}\right) + kv^2\frac{dx}{ds},$$

$$Y = v\frac{d}{ds}\left(v\frac{dy}{ds}\right) + kv^2\frac{dy}{ds},$$

$$Z = v\frac{d}{ds}\left(v\frac{dz}{ds}\right) + kv^2\frac{dz}{ds};$$

which give the component accelerating effects of the force. The force may be more simply found from its component accelerating effects along the tangent and the radius of absolute curvature; for these are respectively

$$v\frac{dv}{ds} + kv^2 \quad \text{and} \quad \frac{v^2}{\rho}.$$

COR. 2. *To determine the motion in a resisting medium of a heavy particle, initially projected in a direction inclined to the vertical.*

It is evident that the particle's motion will be in the vertical plane passing through the line of initial projection. In this plane let the axis of x be drawn horizontally, and the axis of y vertically upwards, through the point of projection. If x, y be the co-ordinates of the particle, and s the length of path described by it at the end of the time t measured from the instant of projection, we have

$$\frac{d^2x}{dt^2} = -k\frac{dx}{dt}\frac{ds}{dt}, \quad \frac{d^2y}{dt^2} = -g - k\frac{dy}{dt}\frac{ds}{dt},$$

g being the accelerating effect of the particle's weight in the medium. From the first equation it appears that

$$\log\left(\frac{dx}{dt}\right) = C - ks.$$

If u be the horizontal component of the velocity of projection, when $s = 0$, $\dfrac{dx}{dt} = u$;

$$\therefore \frac{dx}{dt} = u\epsilon^{-ks}.$$

Hence $\dfrac{dy}{dt} = u\dfrac{dy}{dx}\epsilon^{-ks}$, $\dfrac{d^2y}{dt^2} = u^2\dfrac{d^2y}{dx^2}\epsilon^{-2ks} - ku\dfrac{dy}{dx}\epsilon^{-ks}\dfrac{ds}{dt}$;

the second equation therefore gives, by substitution,

$$\frac{d^2y}{dx^2} + \frac{g}{u^2}\epsilon^{2ks} = 0,$$

the differential equation of the path.

Multiplying by $2\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$ or $2\dfrac{ds}{dx}$, and integrating

$$\frac{dy}{dx}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} + \log\left\{\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right\} + \frac{g}{ku^2}\epsilon^{2ks}$$

$=$ a constant,

viz. $\tan i \sec i + \log(\tan i + \sec i) + \dfrac{g}{ku^2}$, if i be the inclination to the horizon of the initial line of projection. If this equation could be integrated the result would be the equation of the curve described by the particle.

At the highest point of the curve the tangent is parallel to the axis of x; therefore by putting $\dfrac{dy}{dx} = 0$, we get the length of path between the point of projection and the highest point in the curve

$$= \frac{1}{2k}\log\left[1 + \frac{ku^2}{g}\{\tan i \sec i + \log(\tan i + \sec i)\}\right],$$

and the velocity of the particle at the highest point

$$= u\left[1 + \frac{ku^2}{g}\{\tan i \sec i + \log(\tan i + \sec i)\}\right]^{-\frac{1}{2}}.$$

On the one side of the highest point $\dfrac{dy}{dx}$ is positive and on

the other side negative; this appears also from the consideration that the concavity of the curve is always turned downwards. Put $\frac{dy}{dx} = p$ for shortness, and we have

$$\frac{dp}{dx} + \frac{g}{u^2}\epsilon^{2ks} = 0.$$

From this it appears that, as s becomes more and more nearly $= -\infty$, $\frac{dp}{dx}$ becomes more and more nearly zero, or p tends continually to become equal to a constant quantity; in other words, the curve (part of which is the particle's path) as it runs off to infinity on the negative side of the origin tends continually to become a straight line. Hence there is an asymptote to the curve on the negative side of the origin, and if a be the inclination of this asymptote to the horizon, we have

$$\tan a \sec a - \tan i \sec i + \log\left(\frac{\tan a + \sec a}{\tan i + \sec i}\right) = \frac{g}{ku^2}.$$

Again, as s becomes more and more nearly $= +\infty$, $\frac{dx}{dp}$ becomes always more and more nearly zero, and therefore x becomes more and more nearly equal to a constant quantity; that is, the curve, as it runs off to an infinite distance on the positive side of the origin, tends continually to coincide with a straight line parallel to the axis of y. The curve has therefore a vertical asymptote on the positive side of the origin.

Robins, in his *Principles of Gunnery*, concludes from many experiments on military projectiles that the resistance which the air offers to a cannon ball is nearly in the duplicate proportion of the ball's velocity when the velocity does not exceed 1100 feet per second, and that the accelerating effect of the resistance on a 12lb. iron bullet moving with the velocity of 25 feet in a second amounts to about one foot per second. Hence, adopting feet and seconds as units, k for a twelve-pounder is equal to ·0016; and when the twelve-pounder is moving with velocity v the resistance acting on

it has an accelerating effect $= v^2 \times \cdot 0016$. When the bullet moves with a greater velocity than 1100 feet per second, the resistance is greater than this law assigns. The quantity k will of course be different for bullets which differ in size or weight; it will also vary with the state of the air.

If a bullet be projected at a very small angle of inclination to the horizon, the part of its path which lies above the horizontal line through the point of projection may be found approximately. For in this part of the path the tangent at every point will make a small angle with the horizon, and hence, omitting higher powers than the square of p and $\tan i$ in the above equations, we have

$$p - \frac{1}{2k}\frac{dp}{dx} = \tan i + \frac{g}{2ku^2},$$

whence by integration

$$y = x \tan i - \frac{g}{4k^2u^2}(\epsilon^{2kx} - 2kx - 1).$$

The horizontal range of the projectile will be the value of x (different from zero) which satisfies the condition

$$\begin{aligned}\frac{\epsilon^{2kx} - 2kx - 1}{2kx} &= \frac{2ku^2}{g}\tan i \\ &= \frac{k}{g} V^2 \sin 2i;\end{aligned}$$

if V be the velocity of projection. If a table were formed of the numerical values of the function $\dfrac{\epsilon^z - z - 1}{z}$ corresponding to different values of z; then to find approximately the horizontal range of a projectile projected with velocity V at the small inclination i, we should have to find the value of z (n suppose) in the table corresponding to the value $\dfrac{k}{g} V^2 \sin 2i$ of the function, and $\dfrac{n}{2k}$ would be the horizontal range required.

Cor. 3. *To determine the motion in a resisting medium of a particle acted on by a force which tends to a fixed center.*

The particle will move in the plane passing through the center of force and the line of initial projection. Let r, θ be the polar co-ordinates of the particle at the end of the time t, the center of force being pole, and let P be the accelerating effect of the force in direction towards the pole. If kv^2 be the accelerating effect of the resistance on the particle, v being the velocity, the equations of motion are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P - k\left(\frac{ds}{dt}\right)^2 \frac{dr}{ds},$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = -k\left(\frac{ds}{dt}\right)^2 \frac{rd\theta}{ds};$$

ds being an element of path.

From the second of these equations we have

$$\frac{\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)}{r^2\frac{d\theta}{dt}} = -k\frac{ds}{dt}.$$

Hence, integrating and considering k variable for the sake of generality,

$$r^2\frac{d\theta}{dt} = h\epsilon^{-\int k ds};$$

in which h is a constant depending on the initial circumstances of the motion. Again, putting $\frac{1}{u}$ for r

$$\frac{dr}{dt} = -\frac{d\theta}{dt}\frac{du}{d\theta}\frac{1}{u^2} = -h\frac{du}{d\theta}\epsilon^{-\int k ds},$$

$$\frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2}\epsilon^{-2\int k ds} - k\frac{ds}{dt}\frac{dr}{dt}.$$

Substituting in the first equation, it becomes

$$-h^2u^2\frac{d^2u}{d\theta^2}\epsilon^{-2\int k ds} - h^2u^3\epsilon^{-2\int k ds} = -P;$$

$$\therefore \quad \frac{d^2u}{d\theta^2} + u = \frac{P\epsilon^{2\int k ds}}{h^2u^2};$$

the differential equation of the path. The time of describing any arc is to be found from the equation

$$t = \frac{1}{h}\int \frac{1}{u^2}\,\epsilon^{\int k\,ds}\,d\theta$$

taken between the values of θ corresponding to the ends of the arc, as limits; and the velocity is easily found to be

$$= h\epsilon^{-\int k\,ds}\sqrt{u^2 + \left(\frac{du}{d\theta}\right)^2};$$

or $\frac{h}{p}\,\epsilon^{-\int k\,ds}$, if p be the perpendicular on the tangent drawn from the center of force.

66. *To determine the motion in a resisting medium of a particle acted on by given forces and constrained to move along a given smooth curve.*

Let m be the mass of the particle, and let s be the length of path included between a fixed point and the position of the particle at the end of the time t. Let the forces acting on the particle be resolved along the tangent in direction of the motion, along the radius of absolute curvature in direction towards the concavity, and along the perpendicular to the osculating plane; and let S, N, and M denote the component accelerating effects in these respective directions. Lastly, let the force of constraint on the particle, which acts in direction perpendicular to the tangent, have component moving effects R and Q along the radius of absolute curvature and the perpendicular to the osculating plane. If the accelerating effect of the resistance be denoted by $k\left(\frac{ds}{dt}\right)^2$, the equations of motion are,

$$\frac{d^2s}{dt^2} = S - k\left(\frac{ds}{dt}\right)^2, \quad \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 = N + \frac{R}{m}, \quad 0 = M + \frac{Q}{m},$$

ρ being the radius of absolute curvature.

From the first equation we have, (k being supposed constant),

$$\left(\frac{ds}{dt}\right)^2 = 2\epsilon^{-2ks}\int \epsilon^{2ks}\,S\,ds,$$

the proper constant being included in the sign of integration. From the other equations we get

$$R = m\left(\frac{2}{\rho}\epsilon^{-2ks}\int\epsilon^{2ks}S\,ds - N\right), \quad Q = -mM,$$

which give the component moving effects of the force of constraint. The time in which the particle describes any given arc of the path will be,

$$t = \int\frac{\epsilon^{ks}ds}{(2\int\epsilon^{2ks}S\,ds)^{\frac{1}{2}}},$$

taken between limits corresponding to the ends of the arc.

Cor. To find the force which must act on a particle in order that it may move in a resisting medium, in a given manner, in a given curve along which it is constrained to move; let v be the velocity at the extremity of a length of path s measured from a fixed point. The component accelerating effect of the force along the tangent must be

$$= v\frac{dv}{ds} + kv^2,$$

s being supposed to increase with the time. The component accelerating effect of the force perpendicular to the tangent may be any whatever.

67. *To determine the motion in a resisting medium of a particle constrained to move along a given smooth surface and acted on by given forces.*

Let $f(x, y, z) = 0$ be the equation of the surface, and let x, y, z be the co-ordinates of the particle at the end of the time t measured from a fixed instant. Let X, Y, Z be the component accelerating effects, parallel to the respective axes of co-ordinates, of the forces acting on the particle, and let R be the moving effect of the force of constraint which the surface exerts on the particle; this force acts along the normal since the surface is smooth. The equations of motion are

$$\frac{d^2x}{dt^2} = X - k\frac{ds}{dt}\frac{dx}{dt} + \frac{R}{m}\frac{\frac{df}{dx}}{\sqrt{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2 + \left(\frac{df}{dz}\right)^2}},$$

$$\frac{d^2y}{dt^2} = Y - k\frac{ds}{dt}\frac{dy}{dt} + \frac{R}{m}\frac{\frac{df}{dy}}{\sqrt{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2 + \left(\frac{df}{dz}\right)^2}},$$

$$\frac{d^2z}{dt^2} = Z - k\frac{ds}{dt}\frac{dz}{dt} + \frac{R}{m}\frac{\frac{df}{dz}}{\sqrt{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2 + \left(\frac{df}{dz}\right)^2}},$$

in which m is the mass of the particle, and ds an element of the path.

Multiplying these equations in order by $\frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{dz}{dt}$, adding together, and integrating the sum, we have (since $\frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz = 0$, because the path lies along the surface)

$$\left(\frac{ds}{dt}\right)^2 = 2\epsilon^{-2ks}\int \epsilon^{2ks}(Xdx + Ydy + Zdz),$$

which gives the velocity when the integration can be effected.

Multiplying the same equations by $\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}$, in order, adding, and substituting, there results,

$$R = m\left\{\frac{2}{\rho}\epsilon^{-2ks}\int\epsilon^{2ks}(Xdx+Ydy+Zdz) - \frac{X\frac{df}{dx} + Y\frac{df}{dy} + Z\frac{df}{dz}}{\sqrt{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2 + \left(\frac{df}{dz}\right)^2}}\right\},$$

ρ being the radius of curvature in the normal section drawn through ds.

If from the equations of motion R and t be eliminated, the result when integrated will be the equation of a surface which intersects the given surface along the path of the particle.

CHAPTER VII.

THE DISTURBED MOTION OF A PARTICLE.

68. WHEN a particle moves under the action of forces which vary according to different laws of intensity or direction, the equations of motion are generally difficult of solution. In many cases the integrations cannot be effected in finite terms, and consequently the motion can only be approximately determined; in other cases direct integration expresses the result in a form complex and inconvenient. With regard to the motion of a particle, it often happens that if some of the forces acting on the particle were omitted, the motion under the action of the remaining forces would be expressed by equations which admit of ready solution; under such circumstances the method of variable parameters, applied to the solution of the differential equations of motion, may enable us to pass from the more simple to the more complex motion.

Suppose the equations of motion of a particle, whose coordinates at the time t are x, y, z, to be

$$\frac{d^2x}{dt^2} = X + F, \quad \frac{d^2y}{dt^2} = Y + G, \quad \frac{d^2z}{dt^2} = Z + H \ \ldots \ (1)$$

in which X, Y, Z and F, G, H are the component accelerating effects of two sets of forces acting on the particle.

If the set of forces whose component accelerating effects are F, G, H did not act, the equations of the particle's motion would be

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y, \quad \frac{d^2z}{dt^2} = Z \ \ldots\ldots\ldots\ldots\ldots\ldots \ (2)$$

Now let the solution of these equations be

$$x = f(a, \alpha, t), \quad y = g(b, \beta, t), \quad z = h(c, \gamma, t),$$

where f, g, h denote known functions, and $a, \alpha, b, \beta, c, \gamma$ are perfectly arbitrary constants; hence

$$\frac{d^2 f}{dt^2} = X, \quad \frac{d^2 g}{dt^2} = Y, \quad \frac{d^2 h}{dt^2} = Z \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The values of the co-ordinates of the particle which satisfy equations (1) may evidently be expressed in the same form, if the quantities $a, \alpha, b, \beta, c, \gamma$ be considered not constants but functions of t, and so taken that the values of x, y, z expressed in terms of them may satisfy equations (1). The quantities $a, \alpha, b, \beta, c, \gamma$ being in number six, and subject to only three conditions, are therefore, to a certain extent, still arbitrary; they may clearly be subjected to three other conditions. Let them then be so taken that not only the coordinates of the particle whose motion is represented by equations (1) shall be expressed in the same form as if the motion were represented by equations (2), but also the component velocities of the particle shall be expressed in the same form in the one case as in the other:—this gives

$$\frac{dx}{dt} = \frac{df}{dt}, \quad \frac{dy}{dt} = \frac{dg}{dt}, \quad \frac{dz}{dt} = \frac{dh}{dt},$$

and therefore

$$\left.\begin{aligned} \frac{df}{da}\,\frac{da}{dt} + \frac{df}{d\alpha}\,\frac{d\alpha}{dt} &= 0, \\ \frac{dg}{db}\,\frac{db}{dt} + \frac{dg}{d\beta}\,\frac{d\beta}{dt} &= 0, \\ \frac{dh}{dc}\,\frac{dc}{dt} + \frac{dh}{d\gamma}\,\frac{d\gamma}{dt} &= 0, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (4)$$

the differentiations being partial. Hence we have

$$\frac{d^2 x}{dt^2} = \frac{d^2 f}{dt^2} + \frac{d^2 f}{da\,dt}\,\frac{da}{dt} + \frac{d^2 f}{d\alpha\,dt}\,\frac{d\alpha}{dt}$$

and similar expressions for $\dfrac{d^2 y}{dt^2}, \dfrac{d^2 z}{dt^2}$. Substituting these in equations (1), we get {by virtue of equations (3)}

$$\left.\begin{aligned}\frac{d^2f}{da\,dt}\frac{da}{dt}+\frac{d^2f}{d\alpha\,dt}\frac{d\alpha}{dt}&=F\\ \frac{d^2g}{db\,dt}\frac{db}{dt}+\frac{d^2g}{d\beta\,dt}\frac{d\beta}{dt}&=G\\ \frac{d^2h}{dc\,dt}\frac{dc}{dt}+\frac{d^2h}{d\gamma\,dt}\frac{d\gamma}{dt}&=H\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ (5).$$

From the equations (4) and (5) the velocity of change of each of the six parameters $a, \alpha, b, \beta, c, \gamma$ may be immediately found; and each of the parameters must then be got by integration.

If the quantities, $a, \alpha, b, \beta, c, \gamma$ be found in terms of t, and if from the equations

$$x = f(a, \alpha, T), \quad y = g(b, \beta, T), \quad z = h(c, \gamma, T),$$

the quantity T be eliminated, we shall have two equations in x, y, z, and t, which will be the equations of the path along which the particle would proceed after the time t, if at that time and ever after, the forces whose accelerating effects are F, G, H were to cease acting on the particle. The curve so determined at any instant, has evidently the same tangent as the curve which is the actual path of the particle. It is manifest that the form of the curve changes from instant to instant.

Hence the motion of the particle may be regarded as taking place along a curve, the form of which is continually undergoing change; and the changes of form are due entirely to the forces whose accelerating effects are F, G, H. Regarding the motion from this point of view, the curve at any instant is called the instantaneous path of the particle or the undisturbed path, and the forces which cause the changes in the form of the instantaneous path are called disturbing forces. If, at any instant and ever after, the disturbing forces ceased to act, the motion which the particle would have is spoken of as the undisturbed motion; the equations (2) above are the equations of the undisturbed motion.

This way of considering the motion of a particle as being along a curve, the form of which is continually changing by

reason of certain of the forces acting on the particle being accounted disturbing forces, is applicable to constrained motion as well as to free. The only difference will be that the number of arbitrary constants introduced into the undisturbed motion will be less in the case of constrained than in that of free motion. The conditions to which the varying parameters may be subjected will be precisely the same in constrained and free disturbed motion, viz. that the position, velocity, and direction of motion, may be all expressed in the same forms both in the disturbed and undisturbed motions.

When the disturbing forces are very small, the changes in the varying parameters will take place very slowly, and the parameters will for short intervals of time differ but little from absolute constants. In such a case the motion may be determined very approximately during a short interval by considering the parameters invariable in those terms of the differential equations (4) and (5) which involve the accelerating effects of the disturbing forces; this approximation is equivalent to omitting squares and higher powers of the accelerating effects of the disturbing forces.

An important principle in calculating approximately the motion of a particle, disturbed by a number of forces so small that the squares and products of their accelerating effects may be omitted, is that the change in a parameter caused by the joint action of the disturbing forces is equal to the sum of the changes which the forces would severally produce if they acted singly. Let the equation for determining the variation of the parameter a caused by the joint action of the forces whose accelerating effects are f, f', f'' be

$$\frac{da}{dt} = \phi\,(f, f', f''\ldots)$$

ϕ being a function such that $\phi\,(o, o', o''\ldots) = 0$, since if there were no disturbing forces the parameter would be constant. Expanding by Maclaurin's theorem and omitting squares and products of f, f', f'', ... we have

$$\frac{da}{dt} = \frac{d\phi}{do} f + \frac{d\phi}{do'} f' + \frac{d\phi}{do''} f'' + \ldots ;$$

that is, the total variation due to the joint action of the disturbing forces is the sum of the variations due to the separate actions of the forces. If the right hand side of this equation consist of a series of terms which run through all their different values in short periods of time, and a series of terms which are unperiodic; a convenient way of proceeding will be to omit the periodic terms, integrate the unperiodic terms left, substitute the value of the parameter so obtained in the periodic terms, and integrate these terms considering the parameter invariable; the approximate value of the parameter will be the sum of the values obtained from the periodic and unperiodic terms. This process regards the value of the parameter as continually oscillating about a certain mean value determined from the unperiodic terms; and the mean value corrected on account of the oscillations gives the true value. Changes in the mean values of parameters are called secular variations, and changes in the corrections applied to the mean values are called periodic variations. The secular inequalities are not necessarily unperiodic; but, when periodic, their periods are of very much longer duration than those of the periodic inequalities.

69. *To determine the motion of a heavy particle falling from rest, disturbed by the resistance of the medium in which the motion takes place.*

If x denote the vertical distance between the point of initial rest and the position of the particle at the end of the time t, the equation of undisturbed motion is

$$\frac{d^2x}{dt^2} = g,$$

g being the accelerating effect of the weight of the particle in the medium; and the equation of the disturbed motion is

$$\frac{d^2x}{dt^2} = g - k\left(\frac{dx}{dt}\right)^2,$$

the accelerating effect of the resistance being denoted by k (velocity)^2. Now the solution of the former equation is

$$x = a + bt + \frac{gt^2}{2},$$

a and b being constants. Let this be the solution of the latter equation, a and b being variable. By differentiation we have

$$\frac{dx}{dt} = \frac{da}{dt} + \frac{db}{dt}t + b + gt.$$

But if the velocity be expressed in the same form in the disturbed as in the undisturbed motion,

$$\frac{dx}{dt} = b + gt;$$

$$\therefore \frac{da}{dt} + \frac{db}{dt}t = 0.$$

Differentiating the former of these, and substituting in the equation of disturbed motion, we get

$$\frac{db}{dt} = -k(b + gt)^2.$$

From the last two equations a and b are to be found. Adding g to each side of the equation in b and integrating, we have

$$\log\left\{\frac{\sqrt{\frac{g}{k}} + (b + gt)}{\sqrt{\frac{g}{k}} - (b + gt)}\right\} = 2\sqrt{kg}t,$$

the constant being determined by making $b = 0$, when $t = 0$.

Hence

$$b = \sqrt{\frac{g}{k}}\frac{\epsilon^{\sqrt{kg}t} - \epsilon^{-\sqrt{kg}t}}{\epsilon^{\sqrt{kg}t} + \epsilon^{-\sqrt{kg}t}} - gt.$$

Again, from the equation in $\frac{da}{dt}$ we have, integrating by parts,

$$a = -bt + \int b\,dt$$

$$= g\frac{t^2}{2} - t\sqrt{\frac{g}{k}}\frac{\epsilon^{\sqrt{kg}t} - \epsilon^{-\sqrt{kg}t}}{\epsilon^{\sqrt{kg}t} + \epsilon^{-\sqrt{kg}t}} + \frac{1}{k}\log\left(\frac{\epsilon^{\sqrt{kg}t} + \epsilon^{-\sqrt{kg}t}}{2}\right),$$

the constant being determined by making a and t begin together from zero. Thus the motion is determined; the result agrees with that of cor. 2, art. 64.

70. *To determine the motion of a particle, which oscillates in a straight line, under the action of a force, attracting it towards a fixed center with an intensity proportional to its distance from the center, and a disturbing force in the line of motion.*

Let the fixed center be origin of distances, and let x be the distance of the particle from the origin at the time t; let n^2x denote the accelerating effect of the attractive force, and f the accelerating effect of the disturbing force estimated in direction from negative to positive. The equations of the undisturbed and disturbed motions are respectively

$$\frac{d^2x}{dt^2} = -n^2x, \quad \frac{d^2x}{dt^2} = -n^2x + f.$$

The solution of the first of these is

$$x = a\cos nt + b\sin nt,$$

a and b being arbitrary constants. Let this be also the solution of the equation of the disturbed motion, a and b being considered variable. Since the velocity is of the same form in the disturbed as in the undisturbed motion,

$$\frac{dx}{dt} = -na\sin nt + nb\cos nt;$$

$$\therefore \frac{da}{dt}\cos nt + \frac{db}{dt}\sin nt = 0.$$

And by substitution in the equation of disturbed motion we have

$$\frac{da}{dt}\sin nt - \frac{db}{dt}\cos nt = -\frac{f}{n}.$$

From these equations we get

$$\frac{da}{dt} = -\frac{f}{n}\sin nt, \quad \frac{db}{dt} = \frac{f}{n}\cos nt.$$

If the particle be supposed to be initially at rest at distance $a_{\prime}$ from the origin, when $t = 0$, $a = a_{\prime}$ and $b = 0$. Hence,

$$a = a_{\prime} - \frac{1}{n}\int_0^t f \sin ntdt, \quad b = \frac{1}{n}\int_0^t f \cos ntdt,$$

and therefore the distance of the particle from the origin at the time t is

$$= a_{\prime} \cos nt + \frac{1}{n}(\sin nt \int_0^t f \cos nt\,dt - \cos nt \int_0^t f \sin nt\,dt),$$

and the velocity of the particle at the same time is

$$= -na_{\prime} \sin nt + \sin nt \int_0^t f \sin nt\,dt + \cos nt \int_0^t f \cos nt\,dt.$$

If t' denote the time when the particle comes to rest, we have for determining it the equation

$$0 = -na_{\prime} \sin nt' + \sin nt' \int_0^{t'} f \sin nt\,dt + \cos nt' \int_0^{t'} f \cos nt\,dt,$$

or $0 = -na_{\prime} \sin nt' + \int_0^{t'} f \cos n(t' - t)\,dt$,

and the distance from the origin of the point of rest is

$$= a_{\prime} \cos nt' + \frac{1}{n}\int_0^{t'} f \sin n(t' - t)dt.$$

COR. Suppose, for example, that the disturbing force is constant. The velocity of the particle and its distance from the origin at the time t' are respectively equal to

$$-n\left(a_{\prime} - \frac{f}{n^2}\right) \sin nt', \text{ and } \left(a_{\prime} - \frac{f}{n^2}\right) \cos nt' + \frac{f}{n^2}.$$

Hence the particle is at rest at times 0, $\frac{\pi}{n}$, $\frac{2\pi}{n}$, &c., and its distances from the origin at these times are $a_{\prime}$ and $-\left(a_{\prime} - \frac{2f}{n^2}\right)$ alternately. In each oscillation the velocity is greatest at the point whose distance from the origin is $\frac{f}{n^2}$. A disturbing force therefore which acts with constant intensity in the same direction does not affect the time of a vibration; it diminishes the extent of vibration; and shifts the point of greatest velocity in the direction of its action.

If the accelerating effect of the disturbing force be functional of the distance of the particle from the origin or of the velocity, it usually happens that the motion can be only approximately found from the above equations. When the disturbing force is very small, a good approximation will be obtained by expressing the disturbing force's accelerating effect in terms of the parameters and the time, and considering the parameters in the expression constant during a single oscillation.

71. *To determine the oblique motion of a heavy particle disturbed by the resistance of the medium in which the motion takes place.*

Let the axis of x be drawn horizontal and the axis of y vertically upwards in the plane of motion. Let x, y be the co-ordinates of the particle at the end of the time t measured from a fixed epoch. The equations of the disturbed motion are

$$\frac{d^2x}{dt^2} = -k\frac{ds}{dt}\frac{dx}{dt}, \quad \frac{d^2y}{dt^2} = -g - k\frac{ds}{dt}\frac{dy}{dt}$$

if g be the accelerating effect of the particle's weight in the medium, ds an element of the path, and $k\left(\frac{ds}{dt}\right)^2$ the accelerating effect of the resistance of the medium.

Assume that these equations are satisfied by putting

$$x = a + \alpha t, \quad y = b + \beta t - g\frac{t^2}{2},$$

in which a, α, b, β are variable quantities to be determined, and such that if after any instant they be supposed to remain invariable, the values of x and y would satisfy the equations of undisturbed motion

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = -g.$$

Since the component velocities in the disturbed and undisturbed motions are to be expressed in the same forms, we have

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = \beta - gt, \quad \frac{ds}{dt} = \sqrt{a^2 + (\beta - gt)^2};$$

$$\therefore \frac{da}{dt} + \frac{da}{dt}t = 0, \quad \frac{db}{dt} + \frac{d\beta}{dt}t = 0.$$

Again, by substituting in the equation of disturbed motion, we get

$$\frac{da}{dt} = -ka\sqrt{a^2 + (\beta - gt)^2},$$

$$\frac{d\beta}{dt} = -k(\beta - gt)\sqrt{a^2 + (\beta - gt)^2};$$

$$\therefore \frac{da}{dt} = kta\sqrt{a^2 + (\beta - gt)^2},$$

$$\frac{db}{dt} = kt(\beta - gt)\sqrt{a^2 + (\beta - gt)^2}.$$

From these four equations the quantites a, a, b, β are to be found at any time. If the density of the medium be very small, k will be very small, and the changes undergone by the parameters during a short time may be very approximately found by supposing the parameters invariable in the right hand sides of the equations during the short time.

Eliminating t from the equations

$$x = a + at, \quad y = b + \beta t - \tfrac{1}{2}gt^2,$$

we get the equation of the instantaneous orbit, which is a parabola with axis vertical and vertex upwards,

$$\left(x - a - \frac{a\beta}{g}\right)^2 = \frac{2a^2}{g}\left(\frac{\beta^2}{2g} + b - y\right);$$

hence if λ be the latus-rectum of the instantaneous parabola at the end of the time t, and p, q the horizontal and vertical co-ordinates of the focus at the end of the same time,

$$\lambda = \frac{2a^2}{g}, \quad p = a + \frac{a\beta}{g}, \quad q = \frac{\beta^2 - a^2}{2g} + b,$$

and we have

$$\frac{d\lambda}{dt} = \frac{4a}{g}\frac{da}{dt} = -2k\lambda v,$$

putting v for the velocity of the particle,

$$\frac{dp}{dt} = \frac{da}{dt} + \frac{1}{g}\left(\alpha\frac{d\beta}{dt} + \beta\frac{d\alpha}{dt}\right) = -\frac{2k\alpha}{g}(\beta - gt)\,v,$$

$$\frac{dq}{dt} = \frac{db}{dt} + \frac{1}{g}\left(\beta\frac{d\beta}{dt} - \alpha\frac{d\alpha}{dt}\right) = -\frac{kv}{g}(v^2 - 2\alpha^2).$$

From the last of these equations it follows that the focus of the instantaneous parabola moves upwards or downwards, according as the direction of the particle's motion makes with the horizon an angle less or greater than half a right angle; or, in other words, according as the particle is situated above or below the latus-rectum of the instantaneous orbit. From the second of the equations it appears that while the particle is approaching the highest point of the path, the axis of the instantaneous parabola is moving backwards; when the particle is at the highest point, the axis is stationary for the instant; and after the particle has passed the highest point, the axis moves forwards: of course the axis is always vertical. The equation in λ shews that the latus-rectum is continually diminishing; and, by integration, shews that the logarithm of the ratio of the latera-recta in the instantaneous orbit at any two instants is proportional to the length of path described by the particle in the interval between the instants.

72. *A particle, revolving about a fixed center to which it is attracted by a force inversely proportional to the square of the distance, is acted on by disturbing forces in the plane of motion; to find the variations of the elements in the instantaneous orbit.*

Let r, θ be the polar co-ordinates of the particle in the plane of motion at the end of the time t, referred to the fixed center as pole; let $\frac{\mu}{r^2}$ denote the accelerating effect of the central force; and let the disturbing forces have component accelerating effects f along r in direction from the center and f' perpendicular to r in direction tending to increase the angle-vector. The equations of motion are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2} + f, \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = f'$$

The equations of undisturbed motion are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2}, \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0\,;$$

which, by putting $\frac{1}{u}$ for r, may be transformed into (art. 28)

$$\frac{d\theta}{dt} = hu^2, \quad \frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2};$$

h, being a constant. The latter of these equations gives by integration (art. 34)

$$u = \frac{\mu}{h^2}\left\{1 + e\cos(\theta - \varpi)\right\},$$

e and ϖ being constants.

Now since in the instantaneous orbit the velocity and direction of motion are the same as in the disturbed orbit, therefore u, $\frac{du}{d\theta}$ and $\frac{d\theta}{dt}$ are expressed in the same forms, but h, e, and ϖ must be considered variable. From the second equation of disturbed motion we have

$$\frac{dh}{dt} = \frac{f'}{u};$$

$$\text{also } \frac{dr}{dt} = -h\frac{du}{d\theta},$$

$$\text{and } \frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2} - \frac{f'}{u}\frac{du}{d\theta};$$

substituting in the first equation of disturbed motion, we get

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} - \frac{1}{h^2u^2}\left(f + \frac{f'}{u}\frac{du}{d\theta}\right).$$

Again, since $\frac{du}{d\theta} = -\frac{\mu e}{h^2}\sin(\theta - \varpi)$;

$$\therefore\ \frac{d}{d\theta}\left(\frac{1}{h^2}\right) + \frac{d}{d\theta}\left(\frac{e}{h^2}\right)\cos(\theta - \varpi) + \frac{e}{h^2}\frac{d\varpi}{d\theta}\sin(\theta - \varpi) = 0,$$

or, since $\frac{d}{d\theta}\left(\frac{1}{h^2}\right) = -\frac{2}{h^3}\frac{dh}{dt}\frac{dt}{d\theta} = -\frac{2f'}{h^4u^3}$,

$$\frac{d}{d\theta}\left(\frac{e}{h^2}\right)\cos(\theta - \varpi) + \frac{e}{h^2}\frac{d\varpi}{d\theta}\sin(\theta - \varpi) = \frac{2f'}{h^4u^3}.$$

By substitution in the differential equation of the disturbed orbit, we have also

$$\frac{d}{d\theta}\left(\frac{e}{h^2}\right)\sin(\theta - \varpi) - \frac{e}{h^2}\frac{d\varpi}{d\theta}\cos(\theta - \varpi)$$
$$= \frac{1}{h^2u^2}\left\{\frac{f}{\mu} - \frac{f'e}{h^2u}\sin(\theta - \varpi)\right\}.$$

From these last two equations we have, by successive eliminations and multiplying by $\frac{d\theta}{dt}$ or hu^2,

$$\frac{d}{dt}\left(\frac{e}{h^2}\right) = \frac{1}{h}\left[\frac{f}{\mu}\sin(\theta - \varpi) + \frac{f'}{h^2u}\{2\cos(\theta - \varpi) - e\sin^2(\theta - \varpi)\}\right]$$
$$= \frac{1}{\mu h}\left[f\sin(\theta - \varpi) + f'\left\{\frac{\cos(\theta - \varpi) - e}{1 + e\cos(\theta - \varpi)} + \cos(\theta - \varpi)\right\}\right],$$
$$\frac{d\varpi}{dt} = \frac{h}{e}\left[-\frac{f}{\mu}\cos(\theta - \varpi) + \frac{f'\sin(\theta - \varpi)}{h^2u}\{2 + e\cos(\theta - \varpi)\}\right]$$
$$= \frac{h}{\mu e}\left\{-f\cos(\theta - \varpi) + f'\sin(\theta - \varpi)\frac{2 + e\cos(\theta - \varpi)}{1 + e\cos(\theta - \varpi)}\right\}.$$

From these equations and the equation $\frac{dh}{dt} = \frac{f'}{u}$ the elements of the instantaneous orbit at any time are to be found.

If the instantaneous orbit be an ellipse, and a be the semiaxis major, we have

$$h^2 = \mu a(1 - e^2),$$
$$\frac{de}{dt} = h^2\frac{d}{dt}\left(\frac{e}{h^2}\right) + \frac{2e}{h}\frac{dh}{dt},$$
$$\frac{da}{dt} = \frac{2\mu e a^2}{h^2}\frac{de}{dt} + \frac{2a}{h}\frac{dh}{dt}.$$

Hence the equations for determining the variations of the elliptic elements are

$$\frac{da}{dt} = 2e\sqrt{\frac{a^3}{\mu(1-e^2)}}\left[f\sin(\theta-\varpi) + \frac{f'}{e}\{1 + e\cos(\theta-\varpi)\}\right],$$

$$\frac{de}{dt} = \sqrt{\frac{a(1-e^2)}{\mu}}\left[f\sin(\theta-\varpi) + f'\left\{\frac{e+\cos(\theta-\varpi)}{1+e\cos(\theta-\varpi)} + \cos(\theta-\varpi)\right\}\right],$$

$$\frac{d\varpi}{dt} = \frac{1}{e}\sqrt{\frac{a(1-e^2)}{\mu}}\left\{-f\cos(\theta-\varpi) + f'\sin(\theta-\varpi)\frac{2+e\cos(\theta-\varpi)}{1+e\cos(\theta-\varpi)}\right\}.$$

If in these equations f' be put equal to zero, we shall find the effects on the magnitude and position of the instantaneous ellipse produced by a radial disturbing force; by putting $f = 0$, the effects of a transversal disturbing force will be found. The following results may be gathered.

If the only disturbing force be one acting along the radius-vector in direction from the center of force, the major axis and excentricity of the instantaneous ellipse increase while the particle moves from the nearer to the farther apse, and decrease while it moves from the farther to the nearer apse; the apsides revolve in direction opposite to that in which the particle revolves while the latter moves on the side of the latus-rectum on which the nearer apse is situated, and they revolve in the same direction as the particle, while the motion of the latter is on the farther apse-side of the latus-rectum. If the disturbing force act towards the center of force, the major axis and excentricity decrease or increase according as the particle is revolving from the nearer to the farther apse, or from the farther to the nearer apse; and the line of apses revolves in the same direction as the particle or in the opposite direction, according as the particle moves on the nearer or farther apse-side of the latus rectum.

If the only disturbing force be a force acting transversely to the radius vector in direction of the particle's revolution, the major axis of the instantaneous orbit continually increases; the excentricity increases or decreases according as the particle moves on the nearer or father apse-side of two lines

drawn from the center of force and making angles with the less apsidal distance whose common cosine is $\frac{\sqrt{1-e^2}-1}{e}$ (the radical being taken positively); and the apsidal line revolves in the same direction as the particle or in the opposite direction, according as the particle moves from the nearer to the farther apse, or from the farther to the nearer apse. If the disturbing force act in direction contrary to that in which the particle revolves, its effects with respect to the increase or decrease of the major axis and excentricity, and the direction of revolution of the apses are precisely opposite to those of a disturbing force acting in the opposite direction.

Cor. If the disturbing force on the particle be the force of resistance offered by the medium in which the motion takes place, we have

$$f = -kv^2\frac{dr}{ds}, \quad f' = -kv^2\frac{rd\theta}{ds},$$

v being the velocity of the particle, and kv^2 the accelerating effect of the resistance.

$$\therefore f = -kv\frac{dr}{dt} = kvh\frac{du}{d\theta} = -kve\sqrt{\frac{\mu}{a(1-e^2)}}\sin(\theta-\varpi),$$

$$f' = -kvr\frac{d\theta}{dt} = -kvhu = -kv\sqrt{\frac{\mu}{a(1-e^2)}}\{1+e\cos(\theta-\varpi)\}.$$

By substituting these values of f and f', we get for determining the variations of the elliptic elements the equations

$$\frac{da}{dt} = -\frac{2kva}{1-e^2}\{1+2e\cos(\theta-\varpi)+e^2\},$$

$$\frac{de}{dt} = -2kv\{e+\cos(\theta-\varpi)\},$$

$$\frac{d\varpi}{dt} = -\frac{2kv}{e}\sin(\theta-\varpi),$$

in which may be substituted for v its value,

$$v = h\sqrt{u^2+\left(\frac{du}{d\theta}\right)^2} = \sqrt{\frac{\mu\{1+2e\cos(\theta-\varpi)+e^2\}}{a(1-e^2)}}.$$

Hence the major axis of the instantaneous ellipse continually diminishes; the excentricity diminishes when the particle moves on the nearer apse-side of two lines drawn from the center of force and making angles whose common cosine is e with the less apsidal distance, and the excentricity increases when the particle moves on the farther apse-side of these lines; the apsidal line revolves in the direction opposite to the direction of the particle's motion, or in the same direction, according as the particle moves from the nearer apse to the farther, or from the farther apse to the nearer.

MATHEMATICAL WORKS

PUBLISHED BY

JOHN DEIGHTON, CAMBRIDGE.

LECTURES ON PRACTICAL ASTRONOMY. By the Rev. J. CHALLIS, M.A., F.R.S., F.R.A.S., Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge.

A COLLECTION OF PROBLEMS IN ALGEBRAIC GEOMETRY OF TWO DIMENSIONS. By WILLIAM WALTON, M.A., Trinity College.

CORRESPONDENCE OF SIR ISAAC NEWTON AND PROFESSOR COTES, including Letters of other Eminent Men, now first published from the Originals in Trinity College Library: together with an Appendix, containing a variety of other Unpublished Letters and Papers of Newton's. With Notes and Synoptical View of Newton's Life, by the Rev. JOSEPH EDLESTON, M.A., Fellow of Trinity College.

GEOMETRICAL ILLUSTRATIONS OF THE DIFFERENTIAL CALCULUS. By M. B. Pell, B.A., Fellow of St John's College. 8vo. 2*s*. 6*d*.

ON THE LUNAR AND PLANETARY THEORIES; the Figure of the Earth; Precession and Nutation; the Calculus of Variations; and the Undulatory Theory of Optics. By G. B. AIRY, Astronomer Royal. Third Edition. 8vo. 15*s*.

THE PROPOSITIONS IN MECHANICS AND HYDROSTATICS, which are required of Questionists not Candidates for Honours, with Illustrations and Examples, collected from various sources. By A. C. BARRETT, M.A. 8vo. 7*s*.

SOLUTIONS OF THE GEOMETRICAL PROBLEMS proposed at St John's College, Cambridge, from 1830 to 1846, consisting chiefly of Examples in Plane Co-ordinate Geometry. With an Appendix, containing several general Properties of Curves of the Second Order, and the determination of the Magnitude and Position of the Axes of Conic Sections, represented by the General Equation of the Second Degree. By the Rev. THOMAS GASKIN, M.A., late Fellow and Tutor of Jesus College, Cambridge. 8vo. 12*s*.

SOLUTIONS OF THE TRIGONOMETRICAL PROBLEMS proposed at St John's College, Cambridge, from 1820 to 1846. By the Rev. T. GASKIN, M.A. 8vo. 9*s*.

AN ELEMENTARY COURSE OF MATHEMATICS. Designed principally for Students of the University of Cambridge. By the Rev. HARVEY GOODWIN, M.A, late Fellow and Mathematical Lecturer of Gonville and Caius College. Third Edition. 8vo. 18*s*.

MATHEMATICAL WORKS PUBLISHED BY J. DEIGHTON.

A COLLECTION OF PROBLEMS AND EXAMPLES ADAPTED TO THE ELEMENTARY COURSE OF MATHEMATICS. By the Rev. H. GOODWIN, M.A. Second Edition. 8vo. 5s.

SOLUTIONS OF GOODWIN'S PROBLEMS AND EXAMPLES. By the Rev. W. W. HUTT, M.A., Fellow and Sadlerian Lecturer of Gonville and Caius College, Cambridge. 8vo. 8s.

A TREATISE ON THE APPLICATION OF ANALYSIS TO SOLID GEOMETRY. Commenced by D. F. GREGORY, M.A. late Fellow and Assistant Tutor of Trinity College, Cambridge; concluded by W. WALTON, M.A., Trinity College, Cambridge. 8vo. 10s. 6d.

EXAMPLES ON THE PROCESSES OF THE DIFFERENTIAL AND INTEGRAL CALCULUS. By the Rev. D. F. GREGORY. Second Edition, edited by WILLIAM WALTON, M.A. Trinity College, 8vo. 18s.

A TREATISE ON THE DYNAMICS OF A RIGID BODY. By the Rev. W. N. GRIFFIN, M.A. St. John's College, Cambridge. 8vo. 6s. 6d.

SOLUTIONS OF THE EXAMPLES APPENDED TO A TREATISE ON THE MOTION OF A RIGID BODY. By the Rev. W. N. GRIFFIN, M.A. 8vo. 6s.

A TREATISE ON ALGEBRA. By the Rev. GEORGE PEACOCK, D.D., Dean of Ely, Lowndean Professor of Astronomy, &c. &c.
Vol. I. Arithmetical Algebra. 8vo. 15s.
Vol. II. Symbolical Algebra, and its Applications to the Geometry of Position. 8vo. 16s. 6d.

THE ELEMENTS OF THE CALCULUS OF FINITE DIFFERENCES, treated on the Method of Separation of Symbols. By JAMES PEARSON, B.A., Trinity College, Cambridge. Second Edition, Enlarged. 8vo. 5s.

A TREATISE ON THE DIFFERENTIAL CALCULUS. By W. WALTON, M.A. 8vo. 10s. 6d.

A COLLECTION OF PROBLEMS IN ILLUSTRATION OF THE PRINCIPLES OF THEORETICAL MECHANICS. By W. WALTON, M.A. 8vo. 16s.

CONIC SECTIONS: their principal Properties proved Geometrically. By W. WHEWELL, D.D., Master of Trinity College, Cambridge. 8vo. 1s. 6d.

AN ELEMENTARY TREATISE ON MECHANICS, intended for the use of Colleges and Universities. Seventh Edition, with extensive corrections and additions. By W. WHEWELL, D.D. 8vo. 9s.

MECHANICAL EUCLID, containing the Elements of Mechanics and Hydrostatics, demonstrated after the manner of Geometry. By W. WHEWELL, D.D. Fifth Edition. 12mo. 5s.

The Propositions in Mechanics and Hydrostatics, which are required of Questionists not Candidates for Honors, with Illustrations and Examples, collected from various sources. By A. C. BARRETT, M.A. 8vo. 7*s*.

A Familiar Explanation of the Higher Parts of Arithmetic; comprising Fractions, Decimals, Practice, Proportion, and its Applications, &c. &c. With an Appendix. Designed as an Introduction to Algebra. By the Rev. F. CALDER, Head Master of the Grammar School, Chesterfield. 12mo. 3*s*. 6*d*.

A Collection of Aritmetical Questions with Answers; forming a complete set of Examples upon Calder's Arithmetic. 12*s*. 6*d*.

Optical Problems. By A. C. CLAPIN, Bachelor of Arts of St. John's College, Cambridge, and Bachelier-es-Lettres of the University of France. 8vo. 4*s*.

Dynamics, or a Treatise on Motion; to which is added a short Treatise on Attractions. By the Rev. S. EARNSHAW, M.A., St. John's College, Cambridge. Third Edition. 8vo. 14*s*.

A Treatise on Statics, containing the Theory of the Equilibrium of Forces; and numerous Examples illustrative of the general Principles of the Science. By the Rev. S. EARNSHAW, M.A. Third Edition, enlarged. 8vo. 10*s*.

Euclid. (The parts read in the University of Cambridge), from the Text of Dr. Simson, with a large Collection of Geometrical Problems, selected and arranged under the different Books. Designed for the Use of Schools. By the Rev. J. W. COLENSO, M.A., late Fellow of St. John's College, Cambridge, Rector of Forncett St. Mary, Norfolk. 18mo. 4*s*. 6*d*.

Also the above, with a Key to the Problems. 6*s*. 6*d*.

Or, the Geometrical Problems and Key. 3*s*. 6*d*.

Or, the Problems, separately, for Schools where other Editions of the Text may be in use. 1*s*.

Euclid, the Elements of. By R. SIMSON, M.D. Twenty-fifth Edition, revised and corrected by S. MAYNARD. 8vo. 8*s*.; 12mo. 5*s*.

Euclid, the Elements of. From the Text of Simson. Edited in the Symbolical Form, by R. BLAKELOCK. 12mo. 6*s*.

Solutions of the Geometrical Problems proposed at St. John's College, Cambridge, from 1830 to 1846, consisting chiefly of Examples in Plane Co-ordinate Geometry. With an Appendix, containing several general Properties of Curves of the Second Order, and the determination of the Magnitude and Position of the Axes of the Conic Section, represented by the General Equation of the Second Degree. By the Rev. THOMAS GASKIN, M.A., late Fellow and Tutor of Jesus College, Cambridge. 8vo. 12*s*.

Solutions of the Trigonometrical Problems proposed at St. John's College, Cambridge, from 1829 to 1846. By the Rev. T. GASKIN, M.A. 8vo. 9*s*.

An Elementary Course of Mathematics. Designed principally for Students of the University of Cambridge. By the Rev. HARVEY GOODWIN, M.A., late Fellow and Mathematical Lecturer of Gonville and Caius College. Third Edition. 8vo. 18*s*.

An Appendix to the First Edition of an Elementary Course of Mathematics, containing the principal Alterations and Additions introduced in the Second Edition. By the Rev. H. GOODWIN, M.A. 8vo. 1*s*.

A Collection of Problems and Examples adapted to the Elementary Course of Mathematics. By the Rev. H. GOODWIN, M.A. Second Edition. 8vo. 5*s*.

Solutions of Goodwin's Collection of Problems and Examples. By the Rev. W. H. HUTT, M.A., Fellow and Sadlerian Lecturer of Gonville and Caius College, Cambridge. 8vo. 8*s*.

Elementary Chapters in Astromomy from the "Astronomie Physique" of Biot. By the Rev. H. GOODWIN, M.A. 8vo. 3*s*. 6*d*.

A Treatise on the Application of Analysis to Solid Geometry. Commenced by D. F. GREGORY, M.A., late Fellow and Assistant Tutor of Trinity College, Cambridge; concluded by W. WALTON, M.A., Trinity College, Cambridge. 8vo. 10*s*. 6*d*.

Examples on the Processes of the Differential and Integral Calculus. By the Rev. D. F. GREGORY. Second Edition, edited by WILLIAM WALTON, M.A., Trinity College. 8vo. 18*s*.

A Treatise on the Dynamics of a Rigid Body. By the Rev. W. N. GRIFFIN, M.A., St. John's College, Cambridge. 8vo. 6*s*. 6*d*.

Solutions of the Examples appended to a Treatise on the Motion of a Rigid Body. By the Rev. W. N. GRIFFIN, M.A. 8vo. 6*s*.

A Treatise on Optics. By the Rev. W. N. GRIFFIN, M.A. Second Edition. 8vo. 8*s*.

Problems and Theorems of Plane Trigonometry. By the Rev. D. HEWITT, Trinity College, Cambridge. 8vo. 6*s*.

The Elements of Algebra. By the Rev. J. HIND, M.A., late Fellow and Tutor of Sidney Sussex College, Cambridge. Fifth Edition. 8vo. 12*s*. 6*d*.

Introduction to the Elements of Algebra. By the Rev. J. HIND, M.A. Third Edition. 12mo. 5*s*.

Principles and Practice of Arithmetic. By the Rev. J. HIND, M.A. Sixth Edition. 12mo. 4*s*. 6*d*.

Key to Arithmetic. The Solutions of the Questions attended with any difficulty in the Principles and Practice of Arithmetic; with an Appendix, consisting of Questions for Examination in all the Rules of Arithmetic. By the Rev. J. HIND, M.A. 8vo. 5*s*.

Elements of Plane and Spherical Trigonometry. By the Rev. J. HIND, M.A. Fifth Edition. 12mo. 7*s*. 6*d*.

Essay towards an Improved System of Differential Limits. By the Rev. JOSEPH HORNER, M.A., late Fellow of Clare Hall, Cambridge. 8vo. 2*s*. 6*d*.

The Elements of the Conic Sections, with the Sections of the Conoids. By the Rev. J. D. HUSTLER, late Fellow and Tutor of Trinity College, Cambridge. Fourth Edition. 8vo. 4*s*. 6*d*.

Mathematical Tables. By Dr. HUTTON. Edited by OLINTHUS GREGORY. Eleventh Edition. Royal 8vo. 18*s*.

The Elements of Hydrostatics and Hydrodynamics. By W. H. MILLER, M.A., Professor of Mineralogy in the University of Cambridge. Fourth Edition. 8vo. 6s.

Elementary Treatise on the Differential Calculus. By W. H. MILLER, M.A. Third Edition. 8vo. 6s.

A Treatise on Chrystallography. By W. H. MILLER, M.A. 8vo. 7s. 6d.

Elementary Principles of the Theory of Electricity. By the Rev. R. MURPHY, M.A., late Fellow of Caius College, Cambridge. 8vo. 7s. 6d.

The First Three Sections of Newton's Principia, with an Appendix; and the Ninth and Eleventh Sections. Edited by the Rev. JOHN H. EVANS, M.A., late Fellow of St John's College, and Head Master of Sedburgh Grammar School. Third Edition. 8vo. 6s.

Mathematical Tracts:—On La Place's Coefficients; the Figure of the Earth; the Motion of a Rigid Body about its Centre of Gravity; Precession and Nutation. By the Rev. M. O'BRIEN, M.A., Professor of Natural Philosophy, King's College, London. 8vo. 4s. 6d.

An Elementary Treatise on the Differential Calculus, in which the Method of Limits is exclusively made use of. By the Rev. M. O'BRIEN, M.A. 8vo. 10s. 6d.

Treatise on Plane Coordinate Geometry; or the Application of the Method of Coordinates to the Solution of Problems in Plane Geometry. By the Rev. M. O'BRIEN, M.A. 8vo. 9s.

A Treatise on Algebra. By the Rev. GEO. PEACOCK, D.D., Dean of Ely, Lowndean Professor of Astronomy, &c., &c.

Vol. I. Arithmetical Algebra. 8vo. 15s.

Vol. II. Symbolical Algebra, and its Applications to the Geometry of Position. 8vo. 16s. 6d.

Senate-House Problems for 1844. With Solutions, by M. O'BRIEN, M.A., Caius College, and R. L. ELLIS, M.A., Trinity College, Moderators. 4to. *sewed*, 4s. 6d.

Statics (Elementary): or a Treatise on the Equilibrium of Forces in One Plane. 8vo. 4s. 6d.

The Elements of the Calculus of Finite Differences, treated on the Method of Separation of Symbols. By JAMES PEARSON, B.A., Trinity College, Cambridge. Second Edition, Enlarged. 8vo. 5s.

The Theory of Algebraic Equations: A Chapter of Elementary Algebra. By the Rev. W. A. B. THYNNE. 8vo. 6s.

Transactions of the Cambridge Philosophical Society, 8 vols. 4to. with *Plates*. 16*l*. 3s

---

A Collection of Problems in Illustration of the Principles of Theoretical Hydrostatics and Hydrodynamics. By W. WALTON, M.A., Trinity College, Cambridge. 8vo. 10*s*. 6*d*.

A Treatise on the Differential Calculus. By W. WALTON, M.A. 8vo. 10*s*. 6*d*.

A Collection of Problems in Illustration of the Principles of Theoretical Mechanics. By W. WALTON, M.A. 8vo. 16*s*.

The Principles of Hydrostatics: An Elementary Treatise on the Laws of Fluids and their Practical Applications. By T. WEBSTER, M.A., Trinity College, Cambridge. Third Edition. 12mo. 7*s*. 6*d*.

The Theory of the Equilibrium and Motion of Fluids. By T. WEBSTER, M.A. 8vo. 9*s*.

The Elements of Plane Trigonometry, with an Introduction to Analytical Geometry, and numerous Problems and Solutions. By J. M. A. WHARTON. 12mo. 4*s*. 6*d*.

Conic Sections: their principal Properties proved Geometrically. By W. WHEWELL, D.D., Master of Trinity College, Cambridge. 8vo. 1*s*. 6*d*.

An Elementary Treatise on Mechanics, intended for the use of Colleges and Universities. Seventh Edition, with extensive corrections and additions. By W. WHEWELL, D.D. 8vo. 9*s*.